Challenges to Predicative Foundations of Arithmetic

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> The White Rabbit put on his spectacles. "Where shall I begin, please your Majesty?" he asked. "Begin at the beginning," the King said gravely, "and go on till you come to the end: then stop."
>
> Lewis Carroll, *Alice in Wonderland*

This is a sequel to our article "Predicative foundations of arithmetic" (Feferman and Hellman, 1995), referred to in the following as PFA; here we review and clarify what was accomplished in PFA, present some improvements and extensions, and respond to several challenges. The classic challenge to a program of the sort exemplified by PFA was issued by Charles Parsons in a 1983 paper, subsequently revised and expanded as Parsons (1992). Another critique is due to Daniel Isaacson (1987). Most recently, Alexander George and Daniel Velleman (1996) have examined PFA closely in the context of a general discussion of different philosophical approaches to the foundations of arithmetic.

The plan of the present paper is as follows: Section I reviews the notions and results of PFA, in a bit less formal terms than there and without the supporting proofs, and presents an improvement communicated to us by Peter Aczel. Then, Section II elaborates on the structuralist perspective that guided PFA. It is in Section III that we take up the challenge of Parsons. Finally, Section IV deals with the challenges of George and Velleman, and thereby, that of Isaacson as well. The paper concludes with an Appendix by Geoffrey Hellman, which verifies the predicativity, in the sense of PFA, of a suggestion credited to Michael Dummett for another definition of the natural number concept.

I. Review

In essence, what PFA accomplished was to provide a formal context based on the notions of finite set and predicative class and on prima facie evident principles for such, in which could be established the existence and categoricity of

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a natural number structure. The following reviews, in looser formal terms than PFA, the notions and results therein prior to any discussion of their philosophical significance. Three formal systems were introduced in PFA, denoted EFS, EFSC, and EFSC*. All are formulated within classical logic. The language L(EFS), has two kinds of variables:

Individual variables: $a, b, c, u, v, w, x, y, z, \ldots$, and
Finite set variables: $A, B, C, F, G, H, \ldots$

The intended interpretation is that the latter range over *finite sets of individuals.* There is one binary operation symbol (,) for a *pairing function* on individuals, and *individual terms* $s, t, \ldots$ are generated from the individual variables by means of this operation. We have two relation symbols, '=' and '$\in$', by means of which *atomic formulas* of the form $s = t$ and $s \in A$ are obtained. *Formulas* $\varphi, \psi, \ldots$ are generated from these by the propositional operations '$\neg$', '&', '$\vee$', '$\rightarrow$', and by the quantifiers '$\forall$' and '$\exists$' applied to either kind of variable. The language L(EFSC), which is the same as that of EFSC*, adds a third kind of variable:

Class variables: $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \ldots$ [1]

In this extended language, we also have a membership relation between individuals and classes, giving further atomic formulas of the form $s \in \mathbf{X}$. Then formulas in L(EFSC) are generated as before, allowing, in addition, quantification over classes. A formula of this extended language is said to be *weak second-order* if it contains no bound class variables. The intended range of the class variables is the collection of weak second-order definable classes of individuals. We could consider finite sets to be among the classes, but did not make that identification in PFA. Instead, we write $A = \mathbf{X}$ if A and $\mathbf{X}$ have the same extension. Similarly, we explain when a class is a subclass of a set, and so on. A class $\mathbf{X}$ is said to be *finite* and we write Fin($\mathbf{X}$) if $\exists A(A = \mathbf{X})$.

The *Axioms of EFS* are denoted (Sep), (FS-I), (FS-II), (P-I), and (P-II), and are explained as follows: The separation scheme (Sep) asserts that any definable subset of a finite set is finite; that is, for each formula φ of EFS, $\{x \in A \mid \varphi(x)\}$ is a finite set B when A is a given finite set. The axiom (FS-I) asserts the existence of an empty (finite) set, and (FS-II) tells us that if A is a finite set and a is any individual then $A \cup \{a\}$ is a finite set. The pairing axioms (P-I) and (P-II), respectively, say that pairing is one-one and that there is an urelement under pairing; it is convenient to introduce the symbol 0 for an individual that is not a pair.

The *Axioms of EFSC* augment those of EFS by the scheme (WS-CA) for weak second-order comprehension axiom, which tells us that $\{x \mid \varphi(x)\}$ is a class $\mathbf{X}$ for any weak second-order φ. In this language, we allow the formula φ in (Sep) to contain free class variables; then it can be replaced by the assertion that any subclass of a finite set is finite. The following theorem (numbered 1 in PFA) is easily proved by a model-theoretic argument, but can also be given a finitary proof-theoretic argument.

Metatheorem. EFSC is a conservative extension of EFS.

In the language of EFSC, (binary) relations are identified with classes of ordered pairs, and functions, for which we use the letters **f**, **g**, ...,[2] are identified with many-one relations; n-ary functions reduce to unary functions of n-tuples. Then we can formulate the notion of *Dedekind finite class* as being an **X** such that there is no one-one map from **X** to a proper subclass of **X**. By the axiom (Card) is meant the statement that every (truly) finite class is Dedekind finite. The *Axioms of EFSC** are then the same as those of EFSC, with the additional axiom (Card).

Now, working in EFSC, we defined a triple $\langle \mathbf{M}, a, \mathbf{g} \rangle$ to be a *pre-N-structure* if it satisfies the following two conditions:

(N-I) $\quad \forall x \in \mathbf{M}[\mathbf{g}(x) \neq a]$, and

(N-II) $\quad \forall x, y \in \mathbf{M}[\mathbf{g}(x) = \mathbf{g}(y) \rightarrow x = y]$.

These are the usual first two Peano axioms when a is 0 and **g** is the successor operation. By an *N-structure* is meant a pre-N-structure that satisfies the *axiom of induction* in the form

(N-III) $\quad \forall \mathbf{X} \subseteq \mathbf{M}[a \in \mathbf{X} \,\&\, \forall x(x \in \mathbf{X} \rightarrow \mathbf{g}(x) \in \mathbf{X}) \rightarrow \mathbf{X} = \mathbf{M}]$.

It is proved in EFSC that we can define functions by primitive recursion on any N-structure; the idea is simply to obtain such as the union of finite approximations. This union is thus definable in a weak second-order way. From that, we readily obtain the following theorem (numbered 5 in PFA):

Theorem. (Categoricity, in EFSC). Any two N-structures are isomorphic.

Now, to obtain the existence of N-structures, in PFA we began with a specific pre-N-structure $\langle \mathbf{V}, 0, \mathbf{s} \rangle$, where $\mathbf{V} = \{x \mid x = x\}$ and $\mathbf{s}(x) = x' = (x, 0)$; that this satisfies (N-I) and (N-II) is readily seen from the axioms (P-II) and (P-I), respectively. Next, define

$$\text{Clos}^{-}(A) \leftrightarrow \forall x[x' \in A \rightarrow x \in A], \tag{1}$$

and

$$y \leq x \leftrightarrow \forall A[x \in A \,\&\, \text{Clos}^{-}(A) \rightarrow y \in A]. \tag{2}$$

In words, $\text{Clos}^{-}(A)$ is read as saying that A is closed under the predecessor operation (when applicable), and so, $y \leq x$ holds if y belongs to every finite set that contains x and is closed under the predecessor operation. Let

$$\mathrm{Pd}(x) = \{y \mid y \leq x\}. \tag{3}$$

The next step in PFA was to cut down the structure $\langle \mathbf{V}, 0, \mathbf{s}\rangle$ to a special pre-N-structure:

$$\mathbf{M} = \{x \mid \mathrm{Fin}(\mathrm{Pd}(x)) \;\&\; \forall y[y \leq x \rightarrow y = 0 \vee \exists z(y = z')]\}. \tag{4}$$

This led to the following theorem (numbered 8 in PFA):

Theorem. (Existence, in EFSC*). $\langle \mathbf{M}, 0, \mathbf{s}\rangle$ is an N-structure.

To summarize: In PFA, categoricity of N-structures was established in EFSC and existence in EFSC*. Following publication of this work, we learned from Peter Aczel of a simple improvement of the latter result obtained by taking in place of $\mathbf{M}$ the following class:

$$\mathbf{N} = \{x \mid \mathrm{Fin}(\mathrm{Pd}(x)) \;\&\; 0 \leq x\}. \tag{5}$$

Theorem. (Aczel). EFSC proves that $\langle \mathbf{N}, 0, \mathbf{s}\rangle$ is an N-structure.

We provide the proof of this here, using facts established in Theorem 2 of PFA.

(i) $0 \in \mathbf{N}$, because $\mathrm{Pd}(0) = \{0\}$ and $0 \leq 0$.
(ii) $x \in \mathbf{N} \rightarrow x' \in \mathbf{N}$, because $\mathrm{Pd}(x') = \mathrm{Pd}(x) \cup \{x'\}$, and $0 \leq x \rightarrow 0 \leq x'$.
(iii) If $\mathbf{X}$ is any subclass of $\mathbf{N}$ and $0 \in \mathbf{X} \wedge \forall y[y \in \mathbf{X} \rightarrow y' \in \mathbf{X}]$, then $\mathbf{X} = \mathbf{N}$. For, suppose that there is some $x \in \mathbf{N}$ with $x \notin \mathbf{X}$. Let $A = \{y \mid y \leq x \;\&\; y \notin \mathbf{X}\}$; A is finite since it is a subclass of the finite set $\mathrm{Pd}(x)$. Moreover, A is closed under the predecessor operation, and so, A contains every $y \leq x$; in particular, $0 \in A$, which contradicts $0 \in \mathbf{X}$.

The theorem follows from (i)–(iii), since the axioms (N-I) and (N-II) hold on $\mathbf{V}$ and hence on $\mathbf{N}$.

It was proved in PFA that EFSC* is of the same (proof-theoretic) strength as the system PA of Peano axioms and is a conservative extension of the latter under a suitable interpretation. The argument was that EFSC* is interpretable in the system ACA_0, which is a well-known second-order conservative extension of PA based on the arithmetical comprehension axiom scheme together with induction axiom in the form (N-III). Conversely, we can develop PA in EFSC* using closure under primitive recursion on any N-structure. Since any first-order formula of arithmetic so interpreted then defines a class, we obtain the full induction scheme for PA in EFSC*. Now, using the preceding result, the whole argument applies *mutatis mutandis* to obtain the following:

Metatheorem. (Aczel). EFSC is of the same (proof-theoretic)strength as PA and is a conservative extension of PA under the interpretation of the latter in EFSC.

This result also served to answer Question 1 on p. 13 of PFA.

Incidentally, it may be seen that the definition of **N** in (5) above is equivalent to the following:

$$x \in \mathbf{N} \leftrightarrow \forall A[x \in A \ \& \ \mathrm{Clos}^{-}(A) \rightarrow 0 \in A] \ \& \ \exists A[x \in A \ \& \ \mathrm{Clos}^{-}(A)]. \quad (6)$$

For, the first conjunct here is equivalent to the statement that $0 \leq x$, and the second to Fin(Pd(x)). In this form, Aczel's definition is simply the same as the one proposed by George (1987, p. 515).[3] Part of the progress that is achieved by this work in our framework is to bring out clearly the assumptions about finite sets that are needed for it and that are prima-facie evident for that notion.

There is one further improvement in our work to mention. It emerged from correspondence with Alexander George and Daniel Velleman that the remark in footnote 5 on p. 16 of PFA asserting a relationship of our work with a definition of the natural numbers credited to Dummett was obscure. The exact situation has now been clarified by Geoffrey Hellman in the Appendix to this paper, where it is shown that Dummet's definition also yields an N-structure, provably in EFSC.

II. The Structuralist Standpoint and "Constructing the Natural Numbers"

In developing predicative foundations of arithmetic, we have been proceeding from a structuralist standpoint, one that each of us has pursued independently in other contexts. In general terms, structuralism has been described by one of us as the view that "mathematics is the free exploration of structural possibilities, pursued by (more or less) rigorous deductive means" (Hellman 1989, p. 6), along with the claim that,

> In mathematics, it is not particular objects which matter but rather certain 'structural' properties and relations, both within and among relevant totalities. (Hellman 1996, p. 101)

Such general formulations raise questions of scope, for it seems that there must be exceptional mathematical concepts requiring a nonstructural or prestructural understanding so that prior sense can be made of "items *in* a structure," *substructure*, and other concepts required for structuralism to get started.[4] For present purposes, however, this question need not be taken up in a general way, as we may work within a more specialized form of structuralism, one explicitly concerned with number systems. As the other of us has put it:

> The first task of any general foundational scheme for mathematics is to establish the number systems. In both the extensional and intensional approach this is done from the modern *structuralist* point of view. The structuralist viewpoint as regards the basic number systems is that it is not the specific nature of the individual objects which is of the essence, but rather the isomorphism type of the structure of which they form a part. Each structure $\mathcal{A}$ is to be characterized up to isomorphism by a structural property P which, logically, may be of first order or of higher order. (Feferman 1985, p. 48)

So long as this is understood, we may work with a system such as EFSC, leaving open whether this itself is to be embedded in a more general structuralist framework or whether it is thought of as standing on its own.

The central point here is that what we are seeking to define in a predicatively acceptable way is not, strictly speaking, the predicate '*natural number*' *simpliciter*, but rather the predicate '*natural-number-type structure*'. That is, we seek to characterize what it is to be a *structure* of this particular type – what Dedekind (1888) called "*simply infinite systems*" and what set-theorists call "*ω-sequences*" – and also to prove that, mathematically, such structures exist. Once this has been accomplished, we may then, as a *façon de parler*, identify the elements of a particular such structure as "the natural numbers," employing standard numerals and designations of functions and relations, but this is essentially for mathematical convenience. Officially, we *eliminate* the predicate 'is a natural number' in its absolute sense and speak instead of what holds in any natural-number-type structure. And thanks to our (limited) second-order logical machinery, we can render arithmetical statements directly, relativized to structures, as illustrated by the conditions (N-I)–(N-III) (Sec. 1, above); there is no need to introduce a relation of *satisfaction* between structures and sentences.

This standpoint has some implications worth noting. First, since no absolute meaning is being assigned to 'natural number', the same goes for 'nonnumber'. While of course a good definition of 'natural-number-type structure' must rule out anything that does not qualify as such a structure, there is simply no problem of "excluding nonnumbers" such as Julius Caesar (on standard platonist conceptions). This notorious Fregean problem simply does not arise in the structuralist setting. Rather than having to answer the question, "Is Julius Caesar a number?" (and presumably get the right answer), we sidestep it entirely. We even regard it as misleading to ask, "Might Julius Caesar be or have been a number?" for this still employs 'number' in an absolute sense. Of course, Julius Caesar might have been – and presumably is, in a mathematical sense – a member of many natural-number-type structures. On the other hand, we can make sense of standard, mathematically sensible statements such as "3/5 is not a natural number" by writing out "In any structure for the rationals with a substructure for the natural numbers (identified in the usual way), the object denoted '3/5' does not belong to the domain of the latter." And, of course, many elliptical references to "the natural numbers" are harmless.

More significantly, the whole question of circularity in "constructions of the natural numbers" must be looked at afresh. In contrast to 'natural number', 'natural-number-type structure' is an infinitistic concept in the straightforward sense that any instance of such a structure has an infinite domain with (at least) a successor-type operation defined on it. While it might well appear circular to define 'natural number' in terms of a predicate applying to just finite objects – for example, finite sets or sequences from some chosen domain – since it might seem obvious that such objects can do the duty of natural numbers, nevertheless if one succeeds in building up an *infinite structure* of just the right sort from finite objects, using acceptable methods of construction, and then proves by acceptable means that one has succeeded, prima facie one has done as much as could reasonably be demanded.

In predicative foundations, it is quite natural to take the notion 'finite set' as given, governed by elementary closure conditions as in EFSC. The cogency of this can be seen as follows: Within the definitionist framework, a predicatively acceptable domain is one in which each item is specified by a designator, say in a mathematical language. Hence any finite subset of the domain is specificable outright by a disjunction of the form $x = d_1 \vee x = d_2 \vee \cdots \vee x = d_k$, where each d_i is a designator of an object in the domain. Thus, the finite sets correspond to finite lists of designators, and it is reasonable for the definitionist to take *this* notion – "finite list of quasi-concrete objects" – as understood. The claim is, along Hilbertian lines, that this does not depend on a grasp of the *infinite structure of natural numbers*, nor does it depend on an explicit understanding of the even more complex infinite structure of finite subsets ordered, say, by inclusion. Once given such a starting point, the closure conditions of EFSC are then evident.

There is a further related point of comparison between the concepts 'finite set' and 'natural number' that is relevant to our project. Given an infinite domain X of objects, we think of a finite set A of Xs as fully determined by its members. Although certain relations to other finite sets of Xs are also evident for us – for example, adjoining any new element to A yields a finite set – the identity of A *as a finite set* is not conceived as depending on its position in an infinite structure of finite sets of Xs. Yet this "self-standing" character of finite sets is not shared by natural numbers, even on platonist views. To identify a natural number is to identify its position in an infinite structure. Even on a set-theoretic construction, while the sets taken as numbers are of course determined *as sets* by their members, they are not determined *as numbers* until their position in a sequence is determined. Such considerations lead naturally to the structuralist project of PFA.

The significance of these points has perhaps not been sufficiently appreciated because, historically, structuralism has not been articulated independently of platonism. If one succeeds in defining 'natural number' platonistically, say as Frege or Russell did, or as Zermelo or von Neumann did, so that the natural numbers are identified uniquely with particular abstract objects, then, since the whole sequence of natural numbers thus defined together with arithmetic functions and relations are unproblematic as objects in such frameworks, it is a trivial matter to pass to an explicit definition of 'natural-number-type structure': one simply specifies as such a structure any that is isomorphic to the original, privileged one. Then, clearly all the work has gone into the original definition of 'natural number', and questions of circularity are directed there. However, the approach of PFA is different, sharing more with Hilbert's conception of mathematical axioms and reference than with Frege's.[5] For we bypass construction of 'the natural numbers' as particular objects and proceed directly to the infinitistic concept, 'natural-number-type structure' (much as Dedekind [1888] proceeded directly to define 'simply infinite system'). Then, in proving the existence of such structures, we introduce a certain sequence of finite objects available within our framework. Collecting these is predicatively unproblematic, for they are specified as having finitely many earlier elements (including an initial one), not as fulfilling mathematical induction. That they satisfy induction is then proved as a theorem.[6]

Despite this result and the related ones established in PFA – especially the categoricity of our characterization and the proof-theoretic conservativeness of our system over PA – questions have been raised, implicitly and explicitly, concerning circularity and possible hidden impredicativity in the constructions. In the remaining two sections, we will address these specifically.

III. Parsons' Challenge

In his stimulating paper "The Impredicativity of Induction" (I of I in the following), Charles Parsons takes up a number of issues in his typically thoughtful and thorough manner. Our main purpose here is to address the points most directly related as a challenge to what PFA was intended to accomplish, namely, a predicative foundation of the structure of natural numbers, given the notion of finite set of individuals.[7] But it is necessary, first, to make some distinctions in regard to the idea of predicativity. To begin, a putative definition of an object c is said to be *impredicative* if it makes use of bound variables whose range includes c as one of its possible values.[8] Such bound variables may appear attached to quantifiers, or as the variable of abstraction in definitions of sets or functions, or as the variable in a unique description operator, and so on. We do not agree with the position ascribed to Poincaré and Weyl,[9] that impredicative definitions are prima facie viciously circular and to be avoided. For example, we regard the number associated with the Waring problem for cubes – defined as the least positive integer n such that every sufficiently large integer is a sum of, at most, n positive cubes – as a perfectly meaningful and noncircular description of a specific integer; it is known that $n \leq 7$, but beyond that, the exact value of n is not known. While this definition would generally be considered nonconstructive, and is impredicative according to the general idea given above, from a classical predicative point of view it is not viciously circular, since we are convinced by predicative arguments that such a number exists and must have an alternative predicative definition, be it 7 or a smaller integer. So, for us, the issue is to determine when there is a predicative warrant for accepting a prima facie impredicative definition. That cannot be answered without saying what constitutes a predicative proof of existence of objects of one kind or another. Moreover, the above explanation of what it is about the form of a putative definition that makes it impredicative does not tell us what constitutes a *predicative definition*, because it only tells us what should *not* appear in it, and nothing about what (notions, names, etc.) *may* appear in it. Since the latter have to be, in some sense, prior to the object being defined, and since it is not asserted in explaining what is to be avoided just what that is, an answer to this necessarily makes of predicativity a *relative* rather than an *absolute* notion.

Considerations such as this led Kreisel to propose a formal notion of predicative provability *given the natural numbers*, and that was characterized in precise proof-theoretical terms independently (and in agreement with each other) by Feferman (1964) and Schütte (1965). Speaking informally, that characterization takes for granted the notions and laws of classical logic as applied to definitions and statements involving, to begin with, only the natural numbers as the range of bound variables in definitions of sets of natural numbers, and then admits, successively, definitions employing variables for sets ranging over collections of sets that have been comprehended predicatively.[10] The details need not concern us; suffice it to say that Parsons, among others, has found this analysis of predicativity given the natural numbers to be persuasive (I of I, p. 150). However, as he suggests in the latter part of I of I, he also finds it reasonable to ascribe the term 'predicative' to the use of certain generalized inductive definitions that breach the bounds of the Feferman–Schütte characterization. There is no contradiction here from our point of view; the latter simply shifts what the notion of predicativity is taken relative to. One might go further and consider a notion of predicativity relative to the structure of real numbers, if one regarded that structure as well determined, and so on to higher levels of set theory. Though the idea is clear enough, none of these has been studied and characterized in precise proof-theoretical terms.[11]

Now, finally, we return to the program of PFA. There, the aim is to consider what can be done predicatively in the foundations of arithmetic relative to the notion of a finite set of individuals, where the individuals themselves may have some structure as built up by ordered pairs.[12] Philosophically, the significance of this is that we have a prior conception of finite set that does not require the understanding of the natural-number system, and for this notion we have some evident closure principles, which are simply expressed by the axioms (Sep), (FS-I), and (FS-II) of PFA. We do not regard the success of the program PFA to be necessary for the acceptance of the natural-number system, but believe that its success, if granted, is of philosophical interest.

The challenge raised by Parsons in I of I begins with the evident impredicativity of Frege's definition of the natural numbers, in the form

$$\text{(Frege-N)} \qquad Na \leftrightarrow \forall P\{P0 \,\&\, \forall x(Px \rightarrow P(Sx)) \rightarrow Pa\},$$

where the variable P is supposed to range over "arbitrary" second-order entities in some sense or other (Fregean concepts, predicates, propositional functions, sets, classes, attributes, etc.), including, among others, the entity N supposedly being defined. But Parsons enlarges on what constitutes the impredicativity of Frege's definitions in that he says that, to use it to derive induction in the form (say) of a rule,

$$\text{(Ind-Rule)} \qquad \frac{\varphi(0),\, Na \rightarrow [\varphi(a) \rightarrow \varphi(a')],\, Nt}{\varphi(t)}$$

we must allow instantiation of the variable 'P' in (Frege-N) by formulas $\varphi(x)$ which may contain the predicate "N". In this sense, the focus of Parsons' discussion is on the *impredicativity of induction*, rather than the prima facie impredicativity of the putative definition (Frege-N). He expands the implications of this still further as follows:

> The thesis of the present note is that the impredicativity that arises from Frege's attempt to reduce induction to a definition is not a mere artifact of Frege's strategy of reduction. As Michael Dummett observed some years ago, the impredicativity – though not necessarily impredicative second-order logic – remains if we regard induction in a looser way as part of the explanation of the term 'natural number'. If one explains the notion of natural number in such a way that induction falls out of the explanation, then one will be left with a similar impredicativity. (I of I, p. 141; the reference is to Dummett [1978 p. 199].)

Perhaps what we were up to in PFA is orthogonal to the issue as posed in this way by Parsons, but let us see what we can do to relate the two. First, as explained in the preceding section, what we are *not* after is a definition of the notion of natural number in the traditional sense in which this is conceived, but rather it is to establish the existence (and uniqueness, up to isomorphism) of a *natural-number structure*, or N-structure (as it was abbreviated, PFA (i.e., Feferman and Hellman 1995)). Second, induction in the form of the principle (N-III) of Section I, above, is taken to be *part* of what constitutes an N-structure. We agree with Parsons (I of I, p. 145) that "[s]tated as a general principle, induction is about 'all predicates'," but we do not agree with the conclusion that he draws (ibid.) that "[i]nduction is thus inherently impredicative, because ... we cannot apply it without taking predicates involving quantification over [the domain of natural numbers] as instances." Rather, our position is that our – or, perhaps better, Aczel's – proof of the existence (and categoricity) of an N-structure is predicative, given the notion of arbitrary finite set of individuals, and thence in any such structure we may apply induction to any formula that is recognized to define a class in our framework, including formulas that refer to the particular definition of our N-structure. Specifically, within EFSC, these are the weak second-order formulas, in which only quantification over individuals and finite sets is permitted. Of course, if we want to apply induction to more general classes of formulas in our system, or to formulas in more extensive systems, the question of predicativity has to be re-examined on a case-by-case basis. For example, if we expand the system EFSC by a principle that says that in any N-structure we may apply induction to *arbitrary* formulas of L(EFSC), the resulting system EFSC + FI is no longer evidently predicative, given the notion of finite set, but it is so nonetheless. The reason is that EFSC + FI can be interpreted in the system ACA with full second-order induction – which is predicative given the natural numbers according to the Feferman–Schütte characterization. And since, on our analysis, the natural numbers are predicative, given the finite sets, this also justifies EFSC + FI on that same basis. Naturally, one may expect that if the language is expanded by introducing terms for impredicatively defined sets (specified by suitable instances of the comprehension axiom), or if one adds impredicative higher-type or set-theoretical concepts, then the expanded instances of induction that become available will take us beyond the predicative, whether considered relative to the natural numbers or to finite sets.[13] But this cannot be counted as an objection to what is accomplished in PFA. It is not the general principle of induction that is impredicative, but only various of its instances; and those instances that Parsons argues to be impredicative, in the above quotation, are not examples of such, granted the notion of finite set.

Now, finally, and relatedly, we take up the objection that Parsons raises in I of I, pp. 146–68, to the predicativity of Alexander George's (1987) revision of Quine's definition of the natural numbers using quantification over finite sets, which is equivalent to Aczel's definition of an N-structure as we pointed out in Section I, above. Of this he says: "To the claim that the Quinean definition of the natural numbers is predicative, one can also reply that it is so only because the notion of finite set is assumed." Indeed, as the above discussion affirms, we could not agree more. But the reason for his objection then is that "[o]nce one allows oneself the notion of finite set, it seems one should be allowed to use some basic forms of reasoning concerning finite sets," and in particular (according to Parsons) of induction and recursion on finite sets, which would then allow one to define the natural numbers as the cardinal numbers of finite sets. But it is just this that we do *not* assume in EFSC (or EFSC*); no assumptions are made on finite sets besides the closure principles (Sep), (FS-I), and FS-II) (and [Card] in the case of EFSC*). Of course, within our system, once we have an N-structure, we can formally define what it means to be a finite set by saying that it is in one-one correspondence with an initial segment of that structure, and then derive principles of induction and recursion for *that* notion. But we cannot prove that these exhaust the range of the finite-set variables.

IV. The Challenge of George and Velleman

In their 1996 paper, "Two conceptions of natural number" (TC in the following), Alexander George and Daniel Velleman take up the PFA constructions in connection with two main conceptions of natural number, which they describe as "pare down" (PD) and "build up" (BU) corresponding to two ways of characterizing the minimal closure of a set A under an operation f. On the PD approach, this is defined explicitly as the intersection of all sets including A and closed under f. In the case of the natural numbers, this corresponds to the definitions given by Dedekind, Frege, and Russell, essentially as the intersection of all classes containing zero and closed under successor. In contrast, the BU approach provides an inductive definition, illustrated in the case of the natural numbers by clauses such as

(1) 0 is a natural number, and
(2) If n is a natural number, then so is $S(n)$,

together with an *extremal clause*, which says that natural numbers are only those objects generated by these rules. As their discussion brings out, the PD approach comports with a platonist view, according to which impredicative definitions are legitimate means of picking out independently existing sets, whereas the BU approach comports with a constructivist view that rejects the platonist stance and impredicative definitions in favor of rules for generating the intended set of objects. Not surprisingly, neither camp is satisfied with the other's approach, the constructivist rejecting the PD approach as just indicated, but the platonist also rejecting the BU approach as failing properly to define the intended class by failing explicitly to capture the required notion of "finite iteration" of the rules of construction. Furthermore, neither camp is impressed with the other's critique. And so the impasse persists.

The question arises for George and Velleman: To which type of definition should that of PFA be assimilated? As they recognize, it seeks to avoid impredicativity and so surely should not be thought of as a PD definition. On the other hand, in PFA, "the completed infinite" is recognized; moreover (although George and Velleman do not highlight this), an *explicit* definition of "*natural-number-type structure*" is provided, not merely an inductive or recursive description of "natural numbers," and so, assimilation of PFA to the BU approach is misleading. Here we would suggest that a new, third category of definition be recognized, one that combines the explicitness demanded by PD with the predicative methods demanded by BU; it might be called "predicative structuralist" (PS), if one wants a two-letter label. But before recognizing a qualitatively new product, we want to be sure that at least the labeling is honest and accurate.

In notes, George and Velleman raise questions on this score. The essential worry seems to be that the construction in PFA (or its simplification by Aczel) succeeds only if the range of the finite-set quantifiers is restricted to truly finite sets; otherwise, "nonstandard numbers" will not be excluded. But, for some reason, any effort to impose this restriction must appear circular or involve some hidden impredicativity. They put it this way:

> As Daniel Isaacson (1987) suggests, the predicativist definition will be successful only if (i) the second-order quantifier in the definition ranges over a domain that includes all finite initial segments of N and (ii) the domain contains no infinite sets. He concludes that the definition therefore "does not fare significantly better on the score of avoiding impredicativity than the one based on full second-order logic" (p. 156). Feferman and Hellman argue in response (1995, note 5, p. 16) that the existence of the required finite initial segments can be justified predicatively, but it seems to us that they have failed to answer part (ii) of Isaacson's objection, namely that infinite sets must be excluded from the domain of quantification. As we saw earlier, it is this exclusion of infinite sets from the second-order domain that guarantees that Feferman and Hellman's definition will capture *only* natural numbers. In fact, the difficulty here is in effect the same as the difficulty that the platonist finds with the BU definition; it is not the inclusion of desired elements in the domain that causes problems, but rather the exclusion of unwanted elements. (TC, n.9)

Now an adequate response to this requires distinguishing what may be called "external" and "internal" viewpoints concerning formalization of mathematics. From an external standpoint, one views a formalization from the outside and asks whether and how nonstandard models of axioms or defining conditions can be ruled out. Here the metamathematical facts are clear. So long as one works with a consistent formal system based on a (possibly many-sorted) first-order logic, or indeed any logic that is compact, nonstandard models of arithmetic are inevitable. *But this is true even if an impredicative definition of "N-structure" is given.* Even a PD definition in ZFC is subject to this limitation and will have realizations in which "numbers" with infinitely many predecessors appear. No extent of analysis of 'finite' or 'standard number', and so on, can overcome this limitation. What this shows is that the problem of "excluding nonstandard models" in this sense is "orthogonal," so to speak, to the problem of predicativity. All formal definitions are in the same boat, and the only recourse, from the external vantage point, is somehow to transcend the framework of first-order logic. Let us return to this momentarily.

Alternatively, one can look at matters from an *internal* point of view. One accepts the inevitability of nonstandard models of theories built on formal logic, but then one attempts to lay down axioms that are intuitively evident of the informal notions one is trying to capture, and then one seeks to prove the strongest theorems that one can, which, on their ordinary informal interpretation, express interesting and desirable results. Thus, one can lay down closure conditions, as in PFA, that are evident of finite sets, and, although they can hold of other collections as well, the theorems that one proves, such as mathematical induction in specified pre-N-structures, establish desired results even if they can be nonstandardly interpreted. (Bear in mind that every mathematical result about the continuum, say, recovered in ZFC has nonstandard interpretations.) Indeed, on this score, a good case can be made that the predicativist can prove results on the existence and uniqueness of natural-number-type structures that are just as decisive as those the classicist can prove. Let us return to this after elaborating a bit further about what can be said on behalf of PFA and the improvements described in Section I from the *external* viewpoint.

To effect the desired "exclusion of infinite sets" that can lead to "nonstandard numbers," that is, elements of N-structures with infinitely many predecessors, one takes the bull by the horns, so to speak: the exclusion is imposed by fiat in the meta-language by stipulating that we are only concerned with interpretations in which the range of the finite-set quantifiers contains only finite sets. 'Finite' is taken as absolute. This is the framework of "weak second-order logic" in its semantical sense. As is well known, it is noncompact and not recursively axiomatizable, but this is offset by gains in expressive power, exploited in PFA. For now one can collect items of a pre-N-structure that correspond to genuinely finite initial segments of a linear ordering, and this suffices to characterize N-structures.

There is a limited analogy with the classicist's approach via PD definitions, for example, those of Dedekind, Frege, and Russell, formalized say in second-order notation; for these characterize N-structures only if nonstandard, less-than-full ranges of the second-order quantifiers are excluded (so that second-order monadic quantifiers must range over *all* subsets of the domain, precluding Henkin models). The problem of nonstandard models is overcome by moving to noncompact, nonaxiomatizable "full second-order logic." But the analogy is only partial. For, whereas the classical logicist excludes nonfull interpretations on the basis of a claim to understand "*all subsets* of an infinite set," the predicative logicist merely excludes infinite sets from the range of finite-set quantifiers on the basis of a claim to understand '*all finite subsets*'. If the objection is that this is illegitimate because 'finite' "is as much in need of analysis as the concept 'natural number' " (TC, note 9), then it is appropriate to refer back to Section II, above, and the whole case for grounding the infinitistic notion of "natural-number-type structure" on elementary assumptions on finite objects, together with the point made earlier (Sec. III) that nowhere do we have to invoke finite-set induction in order to prove any of our theorems, including the theorem that says that mathematical induction holds in any special pre-N-structure. (*Mutatis mutandis* for the Aczel theorem.) Indeed, since induction is essential to the natural-number concept and to reasoning "about the natural numbers," the very fact that finite-set induction is *not* needed to recover this much counts in favor of the view that 'finite set' is actually *less* in need of analysis than 'natural number'.

Moreover, on the question of existence, there is a fundamental disanalogy between the PD and the PS approaches. For, as George and Velleman bring out, the impredicative definitions of the logicists still must presuppose existence of the minimal closure, and this is an additional assumption, not guaranteed merely by the restriction to full interpretations. There still must *be* some full interpretation of the right sort, that is, containing the real minimal closure. In contrast, the predicative constructions of PFA, Aczel, and the Appendix below yield the desired classes by a restricted comprehension principle, WS–CA. Given finite sets as objects, such a principle is justified much as arithmetical comprehension is; one can even eliminate talk of classes of individuals in favor of satisfaction of formulas, since these contain only bound individual and finite-set variables but no bound class variables.

Thus, the predicative logicist accompanies the platonist classicist only a relatively small step beyond first-order logic; then construction takes over on the new higher ground, while the platonist continues ascending, eventually into the clouds.

Consistently with this external view, one can, however, also pursue the internalist course of proving desirable theorems. Here, perhaps surprisingly, the predicativist is able to recover predicativist analogues of well-known classical results. The proofs of categoricity or unicity of N-structures and of mathematical induction in the pre-N-structures of PFA, Aczel, and the Appendix already illustrate this. But one can go further and prove theorems that, informally understood, say explicitly that N-structures cannot contain any nonstandard elements. The idea is to formalize the following, familiar reasoning. Let $\langle \mathbf{M}, 0, ' \rangle$ be an N-structure. Induction implies than any non-empty class (subclass of **M**) closed downward under predecessor, $p(x)$, contains 0. Consider the class of nonstandard numbers (of **M**); call it **K**. If $z \in \mathbf{K}$, then also $p(z) \in \mathbf{K}$ (contraposing the Adjunction axiom); therefore, if **K** is non-empty, it contains 0, a blatant contradiction. (Put positively, 0 is standard and if z is standard, so is z', and so, all members of **M** are standard.) Elements with infinitely many predecessors are ruled out directly by Induction.

But in what system is the above reasoning carried out? If we attempt to formalize it in EFSC, expressing "x is nonstandard" by "$\forall A(A \neq \{y : y \leq x\}$," we immediately contradict the definition of **M**! On the other hand, we cannot simply plug in "$\{y : y \leq x\}$ is Dedekind-infinite" or any other second-order analysis of "infinite" involving general class or function quantifiers, for then we would not be able predicatively to form the class **K**. However, there is an alternative method that gets around this. For here we may appeal to the metatheorem mentioned in Section III: If we add to EFSC the axiom schema known as "full induction" (FI), that is, induction for *arbitrary* second-order formulas, the resulting system, EFSC + FI, is interpretable in the subsystem of PA^2 known as ACA. This also contains FI and, moreover, is a predicatively acceptable system relative to the natural numbers (on the Feferman–Schütte characterization) as noted in Section III. But, as was also observed there, since the natural numbers or N-structures are predicative given the finite sets, EFSC + FI is also predicatively acceptable relative to the finite sets. Although it cannot prove the existence of subclasses of an N-structure defined by formulas with class quantifiers, it can prove that induction holds directly for any formula that platonistically defines a subclass, as it were. In particular, now one can formalize the above induction ruling out nonstandard numbers, using, in place of $x \in \mathbf{K}$, a second-order formula $\varphi(x)$ to express "x has infinitely many predecessors," for example, "the predecessors of x form a class, a subclass of which is in one-one correspondence with an unbounded subclass of **M**"; or it could just as well be "the predecessors of x form a Dedekind-infinite class." The predicativist, as well as the classicist, regards these as good formalizations of the intended notion. Thus, the reasoning is formalizable in a predicatively acceptable extension of EFSC without appealing to the special finite-set variables and without any circular or impredicative reference to the class **K**.[14]

Looking at the contrapositive, one sees that one has thus *derived* the consequence of the axiom (Card) directly relevant to ruling out nonstandard members of N-structures, viz., the statement that the predecessors of any such element form a Dedekind-finite set,

$$\forall x[x \in \mathbf{M} \rightarrow \text{Ded Fin}(\text{Pd}(x))].$$

This follows straightforwardly by induction on the formula, $\varphi(z)$, expressing Ded Fin(Pd(z)). Again, we need not be able to collect all elements satisfying this formula in order to reason with it by mathematical induction.

Thus, "nonstandard numbers" are ruled out as decisively as they can be. From the external standpoint, they are excluded by the semantics of weak second-order logic, which, as has been argued, is a good framework for elementary predicative mathematics. From an internal perspective, without falling back on special finite-set variables, we can employ standard, logicist *analyses* of 'finite', 'infinite', and so forth, and derive theorems in predicatively acceptable systems that directly express the desired exclusion. This may seem like "having one's cake and eating it at the same time." But really it is more like having two desserts.

NOTES

1. The class variables are given in boldface, to distinguish them from the finite set variables.
2. As a point of difference with PFA, function variables here are given in boldface to indicate that they are treated as special kinds of classes.
3. That, in turn, was a modification of a definition of the natural numbers proposed by Quine (1961) using only the first conjunct in (6), which is adequate when read in strong second-order form, but not when read in weak second-order form; cf. George (1987, p. 515) and George and Velleman (1996, n.10).
4. For a good discussion of this and related issues, see Parsons (1990).
5. For a valuable discussion of Hilbert's structuralist views of axioms and reference in mathematics and the contrast with Frege's views, see Hallett (1990).
6. Our construction thus improves on Dedekind's, for he relied, for a Dedekind-infinite system, on a totality – of "all things which can be objects of my thought" (Dedekind 1888, Theorem 66) – which, even apart from its unmathematical character, is unacceptable to a predicativist on logical grounds, for, presumably, such a totality would contain itself! Furthermore, for a *simply infinite system*, he then relied on a subtotality impredicatively specified as the intersection of all subtotalities containing an initial element and closed under the given function (1888, Theorems 72 and 44). But it is noteworthy that the particular example that Dedekind sought to invoke to ensure nonvacuity of his definitions was not identified as "the numbers." As it happened, Dedekind did go on to speak of such abstract particulars, but that is another story, and, in any case, it is a further move that we have not been tempted to make.
7. Parsons' paper appeared well before PFA, and so, the challenge was not issued to *it* but rather to the kind of program that it exemplifies. That challenge was addressed briefly in the final discussion section of PFA, pp. 14–15, but is expanded on substantially here.
8. The informal explanation of what constitutes an impredicative definition varies from author to author. A representative collection of quotations is given by George (1987); the explanation given in the text here is closest to that taken by George from an article of Hintikka (1956).
9. Cf. I of I, pp. 152–3 and p. 159, n.24.
10. To be more precise, this is spelled out by means of an autonomous transfinite progression of ramified systems, where autonomy is a bootstrap condition that restricts one to those transfinite levels that have a prior predicative justification; cf. Feferman (1964).
11. The relative notion of predicativity is recast by Feferman (1996) in terms of a formal notion of the *unfolding* of a schematic theory, which is supposed to tell us what more should be accepted once we have accepted basic notions and principles.
12. Parsons has an interesting discussion in I of I (pp. 143–5), of what is reasonable to assume about the range of first-order variables in proposed definitions of the natural numbers. We believe that the assumptions (P-I) and (P-II) are innocuous, in the sense that the notion of ordered pair is a prerequisite to an understanding of any abstract mathematics.
13. Addition of higher types or even set-theoretical language does not per se force us into impredicative territory; cf. Feferman (1977).
14. There is some irony in the fact that George and Velleman, after claiming (TC, note 10) that the Aczel construction cannot rule out nonstandard numbers without a circular appeal to "the complement of N," present an argument of their own for

the predicative acceptability of an extension of EFSC* in which the full induction schema is derivable. (See their note 14.) They argue for a direct extension to include the separation schema for finite sets with arbitrary second-order formulas. This is closely related to the fact that, in a weak subsystem of analysis, FI is equivalent to the so-called "bounded comprehension scheme,"

$$\forall n \exists X \forall m (m \in X \leftrightarrow m < n \,\&\, \varphi(m)),$$

where $\varphi(m)$ is any formula of second-order arithmetic (lacking free 'X'). (See Simpson [1985, p. 150].) This corresponds to the separation scheme for finite sets with arbitrary second-order formulas. We prefer the direct route to full induction via ACA and proof theory, since it is predicatively problematic to say that an arbitrary formula "specifies unambiguously which elements of the [given] finite set are to be included in a subset" (TC, note 14). It then turns out that their proposed stronger separation scheme is derivable from full induction, and so inherits a predicative justification after all. In any case, once full induction is available, the reasoning that N-structures are truly standard is predicatively formalizable without appeal to finite-set variables, even while employing a standard logicist analysis of 'finite' or 'infinite' as just indicated.

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Appendix: Realizing Dummett's Approach in EFSC

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Here the notation of Section I will be followed, including the use of F, G, H, as finite-set variables. Let 0 and $'$, respectively, be the initial element and successor-like function of a given pre-N-structure. Our principal aim is to prove the following:

Theorem. (EFSC):
Let $\mathbf{M} =^{df} \{x : \exists F(0 \in F \,\&\, \forall y(y \in F \,\&\, y \neq x \rightarrow y' \in F) \,\&\, x \in F \,\&\, \forall G[[(0 \in G \,\&\, \forall y(y \in G \,\&\, y \neq x \rightarrow y' \in G)\} \rightarrow F \subseteq G])\}$, that is, x belongs to a (the) minimal finite set containing 0 and "closed upward except at x." Then $\langle \mathbf{M}, 0, ' \rangle$ is an N-structure.

Remark: Note that this definition of **M** incorporates both an existential condition and a universal one, corresponding to the conditions Wang attributes to Dummett (Wang 1963; cf. TC, note 10).

The proof is simplified by adopting the following abbreviations, which also bring out the relationship between this theorem and that of Aczel:

$$\text{Clos}_d(F, [z, x]) \equiv z \in F \,\&\, \forall y(y \in F \,\&\, y \neq x \rightarrow y' \in F),$$

read as "F is closed upward from z to x." (The subscript d is for Dummett.) Next define

$$z \leq_d x \text{ by } F \equiv \text{Clos}_d(F, [z, x]) \,\&\, x \in F \,\&\, \forall G(\text{Clos}_d(G, [z, x]) \rightarrow F \subseteq G).$$

This can be read as "F witnesses $z \leq_d x$." Trivially, if both F_1 and F_2 witness $z \leq_d x$, then $F_1 = F_2$ (extensionally). Now, define

$$z \leq_d x \equiv \exists F(z \leq_d x \text{ by } F).$$

Now, **M** in the Theorem can be defined by $\mathbf{M} = \{x : 0 \leq_d x\}$. For purposes of comparison, recall the Aczel construction, $\mathbf{M} = \{x : \text{Fin}(\text{Pd}(x)) \,\&\, 0 \leq x\}$, where '$\leq$' is the ordering introduced in PFA, as in Section I, above. We now proceed to the proof of the theorem.

Proof: Let F_x denote the unique F that witnesses $0 \leq_d x$. Then, to say that $x \in \mathbf{M}$ is to say that F_x exists. We have

(i) $0 \in \mathbf{M}$, to wit $\{0\}$ as F_0.

(ii) $z \in \mathbf{M} \rightarrow z' \in \mathbf{M}$.

Given F_z, set $F_{z'} = F_z \cup \{z'\}$. We must show that this works. $F_z \cup \{z'\}$ is finite, by adjunction (FS-II). $0 \in F_z \cup \{z'\}$ and $z' \in F_z \cup \{z'\}$. Now if $y = z$, then trivially $y' \in F_z \cup \{z'\}$; and if $y \neq z$, then if $y \in F_z \cup \{z'\}$ & $y \neq z'$, then $y \in F_z$, and by hypothesis then so is y', whence $y' \in F_z \cup \{z'\}$. Thus, $\mathrm{Clos}_d(F_z \cup \{z'\}, [0, z'])$. It remains to prove minimality.

Suppose $\exists u \in F_z \cup \{z'\}$ such that $u \notin G$, some G such that $\mathrm{Clos}_d(G, [0, z'])$. Consider $G - \{z'\}$. If $y \neq z$ & $y \in G - \{z'\}$, $y \neq z'$ either, so $y' \in G - \{z'\}$ by hypothesis on G. So, $\mathrm{Clos}_d(G - \{z'\}, [0, z])$, and so, $F_z \subseteq G - \{z'\}$, by hypothesis on F_z. Thus, $z \in G - \{z'\}$, and so, $u \neq z'$, by the closure condition on G that forces $z' \in G$. Therefore, $u \in F_z$ but, by hypothesis that $u \notin G$, $u \notin G - \{z'\}$ either, contradicting the minimality of F_z. This completes the proof of minimality of $F_z \cup \{z'\}$ and of step (ii).

(iii) Induction, N-III: Let $\mathbf{X}$ be a class such that $0 \in \mathbf{X}$ and $y \in \mathbf{X} \rightarrow y' \in \mathbf{X}$, all $y \in \mathbf{M}$.

To prove: $z \in \mathbf{M} \rightarrow z \in \mathbf{X}$.

We will prove $F_z \subseteq \mathbf{X}$, which suffices since $z \in F_z$ and indeed $z \in \cap[G : \mathrm{Clos}_d(G, [0, z])]$. Let $H = G \cap \mathbf{X}$, for some such G (existence by F_z itself). H is finite by WS-Sep. We have $\mathrm{Clos}_d(H, [0, x])$ by the closure conditions on G and X. Therefore, $F_z \subseteq H = G \cap \mathbf{X}$, whence $F_z \subseteq \mathbf{X}$. ■

By virtue of the unicity of N-structures ("categoricity," Theorem 5 of PFA), an N-structure can be represented as of the form $\langle \mathbf{M}, 0, ' \rangle$ of Theorem 1, that is, the domain $\mathbf{N}$ of any N-structure $= \{x \in \mathbf{N} : 0 \leq_d x\}$, where 0 here is the initial element of $\mathbf{N}$ and $\leq_d$ is defined over $\mathbf{N}$ via the successor-type relation on $\mathbf{N}$.

As expected, the ordering $\leq_d$ is closely related to '$\leq$' of PFA. This is spelled out in the following.

Theorem. (EFSC):

(1) In any pre-N-structure,

$$z \leq_d x \rightarrow z \leq x.$$

(2) In any N-structure $\langle \mathbf{M}, 0, ' \rangle$ defined as in Theorem 1, and hence in any N-structure,

$$z \leq_d x \leftrightarrow z \leq x.$$

Proof: (1) and $\rightarrow$ of (2): Suppose the implication fails, that is, that $z \leq_d x$ but $\exists A(x \in A \;\&\; \forall y(y' \in A \rightarrow y \in A) \;\&\; z \notin A)$. (So, $z \neq x$, and $z' \notin A$, nor is z'', etc.) Let F be the witness to $z \leq_d x$; that is, $F = \cap[G : \mathrm{Clos}_d(G, [z, x])]$.

Let $B = \{u : u \notin A \,\&\, u \in F\}$. B is finite, by WS-Sep, and $z \in B$ and $\forall y(y \in B \,\&\, y \neq x \rightarrow y' \in B)$; that is, $\mathrm{Clos}_d(B, [z, x])$. So, by definition of $z \leq_d x$, we have $x \in B$, that is, $x \notin A$, a contradiction. (Remark: Note the similarity to the Aczel proof, except that here we are "stepping forward" instead of "stepping back".)
$\leftarrow$ of (2): Now assume that we are in an N-structure, $\langle \mathbf{M}, 0, ' \rangle$, as in Theorem 1. We proceed by induction on z:

(i) For $z = 0$, the implication is trivial.
(ii) Let $z = y$. If $y = x$, the implication is trivial, as then $y' \not\leq x$. Let $y \neq x$, and let H_y be the (minimal) witness to $y \leq_d x$, which we can suppose, by inductive hypothesis. We claim that $H_y - \{y\}$ is the minimal witness to $y' \leq_d x$. Since $y \in H_y$ and $y \neq x$, $y' \in H_y$ by $\mathrm{Clos}(H, [y, x])$, and so, $y' \in H_y - \{y\}$. If $u \in H_y - \{y\}$ and $u \neq x$, then, because $u \neq y$, $u \in H_y$, and so, $u' \in H_y$, whence $u' \in H_y - \{y\}$. (In the last step, we appeal to the minimality of H_y, which implies that $p(y) \notin H_y$, so that $u' \neq y$.) It remains to prove minimality of $H_y - \{y\}$ as witness to $y' \leq_d x$. Let G be such that $\mathrm{Clos}_d(G, [y', x])$, and suppose $\exists u$ such that $u \in H_y - \{y\}$ but $u \notin G$. Then, $u \in H_y$ but $u \neq y$, so $u \notin G \cup \{y\}$. But, $G \cup \{y\}$ contains y and meets the closure condition for H_y, viz. $\mathrm{Clos}_d(G \cup \{y\}, [y, x])$. Therefore, by minimality of H_y, $H_y \subseteq G \cup \{y\}$, contradicting the supposition of u. This proves the minimality of $H_y - \{y\}$ and completes the inductive step. ■

Thus, the Dummett-inspired construction of Theorem 1, as well as the Aczel construction, defines N-structures, provably in EFSC. And the orderings involved, $\leq_d$ and $\leq$, respectively, are extensionally equivalent in these structures.