Does Category Theory Provide a Framework for Mathematical Structuralism?[*]

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Abstract

Category theory and topos theory have been seen as providing a structuralist framework for mathematics autonomous *vis à vis* set theory. It is argued here that these theories require a background logic of relations and substantive assumptions addressing mathematical existence of categories themselves. We propose a synthesis of Bell's "many-topoi" view and modal-structuralism. Surprisingly, a combination of mereology and plural quantification suffices to describe *hypothetical large domains*, recovering the Grothendieck method of universes. *Both* topos theory *and* set theory can be carried out relative to such domains; puzzles about "large categories" and "proper classes" are handled in a uniform way, by relativization, sustaining insights of Zermelo.

Contents

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1 Introduction

In previous work, we have compared three varieties of mathematical structuralism, which we called "set-theoretic", "*sui generis*", and "modal".[15] It was noted that a fourth variety based on category theory deserves comparable, systematic consideration. This paper is aimed at providing that.

The suggestion that category theory provides a way of realizing structuralism as a foundational framework and philosophical interpretation of mathematics can be found in writings of Mac Lane, Moerdijk, Bell and others, and recently Awodey. Having described how category theory provides a systematic notion of mathematical structure based essentially on families of structure-preserving mappings, Awodey summed up:

> "The structural perspective on mathematics codified by categorical methods might be summarized in the slogan: The subject matter of pure mathematics is invariant form, not a universe of mathematical objects consisting of logical atoms...My aim here was not to make the case for philosophical structuralism, but to suggest that it be pursued using a technical apparatus other than that developed by logical atomists since Frege, one with a mathematical heritage sufficiently substantial, and mathematical applications sufficiently uniform, to render significant a view of mathematics based on the notion of 'structure'." ([1], pp. 235-6)

This is an intriguing suggestion. It is naturally viewed in the context of Mac Lane's repeated claim that category theory provides an autonomous foundation of mathematics as an alternative to set theory. The reason for this should be clear: if category theory is not autonomous but rather must be seen ultimately as developed within set-theory, then Awodey's suggestion could not be realized, at least not on the standard way of reading set-theory, viz. as axiomatizing central truths about "the cumulative hierarchy" (presumably consisting of sets as "logical atoms"). Thus, we cannot hope to assess Awodey's suggestion without also (re)examining Mac Lane's thesis.[1]

It must be stressed, however, that we are decidedly *not* attempting to evaluate category theory *as mathematics*. Our concern is to examine

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[1]For an important critique of Mac Lane's thesis, see [9]. More will be said about this below.

its credentials as a framework for a *structuralist understanding* of mathematics, without in any way calling into question the mathematical value or interest of its conceptual machinery or its theorems. Moreover, we are not even trying to assess its overall foundational contribution. For example, claims on its behalf to provide a kind of conceptual unification through functorial relations not (readily) achieved in set theory certainly have foundational significance, but only insofar as they bear directly on the issues of structuralism are they at issue here.

The rest of this paper is organized as follows: The next section scrutinizes the formulation of category theory, distinguishing between its role as pure mathematics and its alleged role as providing a foundation. Section 3 pursues the latter further, examining Mac Lane's proposal to reconstruct mathematics "in a well-pointed topos". An early critique of Feferman is reexamined and reinforced in light of the purely definitional role of "structural axioms". Category and topos theory are found wanting both a prior, external theory of relations as well as substantive axioms of mathematical existence. In section 4, we take up Bell's "many-topoi" view, clarifying the sense in which mathematics becomes "relative to a topos", but finding that the diagnosis of section 3 is essentially applicable. Taking up Bell's suggestion to treat topoi as "possible worlds for mathematics", we examine in section 5 the problem of integrating category theory in a modal-structural framework, which would supply both the wanted prior theory of relations and substantive existence postulates at once (in the form of mathematico-logical possibility claims). The details of this are developed in sections 6, in a sketch of a "theory of large domains", inspired by earlier work of Lewis in connection with set theory. The conclusion sums up some of the main implications of our proposal.

2 How Is Category Theory Formulated?

If one looks at Mac Lane and Moerdijk [27] (cited occasionally below as 'M&M'), one might naturally say, "informally, like other branches of mathematics", as one finds in typical texts. Now everyone knows that analysis, say, can be framed entirely in set-theoretic terms, and it is enough to allude to this fact, as nothing of interest in analysis proper is to be gained by repeating well-known constructions from the ground up. (Similarly for algebra, pure geometry, topology, etc.) In ordinary mathematics texts, the informal presentation is "unofficial"; officially, the framework is set theory. But Mac Lane is famous for proposing category theory as capable of serving as an autonomous foundation, and so it may be surprising to read early on that "we shall not be very explicit about set-theoretic foundations, and we shall tacitly assume we

are working in some fixed universe U of sets."([27], 12.) It is then explained how the distinction between "small" and "large" categories is made, relative to U. Set-theoretically speaking, if U itself is a set, then large categories relative to U can become small in a proper extension of U. (Even here, I am already going beyond what M&M say explicitly. Nothing of concern to their development of category and topos theory as mathematics turns on just how U is chosen.) If this *is* the official position, however, then category theory is to be viewed as part of set theory after all, in which case it is not an autonomous vehicle for articulating structuralism or anything else. Of course, as such it may still contribute its own characteristic notions of "structure", "structure-preserving map", "isomorphism", etc. But all of this would be relative to a given set-theoretic background, and none of the problems affecting set-theoretic structuralism, as described for instance in [15], would be resolved or avoided. We will therefore assume that, despite what M&M imply here, we are to seek an alternative formulation in the interests of an autonomous framework.

Indeed, when it comes to specifically foundational questions, M&M do suggest an alternative: mathematics generally can be reformulated "topos-theoretically" instead of set-theoretically. A topos, $\mathcal{E}$, characterized by "axioms of topos theory" can itself serve as a "universe of discourse", much as a set-theoretic universe does, although for classical mathematics, the first-order, elementary topos axioms must be supplemented so that Boolean, classical logic will be available. (Without supplementation, which M&M specify in detail, the underlying logic of an elementary topos will be intuitionistic.) The fundamental primitive notions for mathematics then become those of the topos axioms, namely "composition" (implicitly of functions or morphisms), "domain", and "co-domain", as governed by the axioms on "arrows in a well-pointed topos". In fact, M&M take pains to distinguish two ways of "working in a topos $\mathcal{E}$": one they call "internal", which appeals only to the topos axioms free of any set-theoretic assumptions, "viewed as (part of) an independent description of a category $\mathcal{E}$ as a universe of discourse". ([27], 235.) In contrast, the "external" view treats topos theory set-theoretically, as suggested in the above-cited preliminary remarks. Clearly, it is the internal perspective that offers the prospect of an autonomous framework. Indeed, as M&M show, a powerful set theory (although not all of ZFC) itself gets "recovered" in topos-theoretic terms, and this by itself already lends some initial credibility to the foundational claims made on behalf of topos-theory. These will have to be scrutinized carefully below.

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3   Mathematics "in a well-pointed topos"?

An elementary topos is a category meeting certain structural conditions supporting certain operations called finite limits, exponentiation, and subobject classification. (For technical details, see [27], Ch. IV, also [26], 398 ff.) The further condition of "well-pointedness" guarantees that functions $f, g : A \to B$ differ just in case $fx \neq gx$ for some "global element" x (an arrow from terminal object 1 to A).[2] This guarantees enough global elements to test diagrams for commutativity, and insures extensional discrimination of functions as in classical set theory. In addition, axioms are added guaranteeing a "natural number" object ("object" in the sense of category theory, i.e. what would normally be called a natural number structure!), and a version of the axiom of choice.[3] Given these conditions, as Mac Lane puts it,

> It is now possible to develop almost all of ordinary Mathematics in a well-pointed topos with choice and a natural number object...The development would seem unfamiliar; it has nowhere been carried out yet in great detail. However, this possibility does demonstrate one point of philosophical interest: The foundation of Mathematics on the basis of set theory (ZFC) is by no means the only possible one! ([26], 402.)

Now, I will not question the claim concerning the strength of topos theory; but there is one respect in which the second statement about unfamiliarity may seem puzzling. After all, one of the interesting facts about category theory is its ability to recover (a generalization of) the notion of elementhood, the set-theoretic primitive. (See e.g. [1], 221 f.) Moreover, as M&M show in some detail, topos theory has the strength

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[2] An object o in a category is *terminal* just in case for each object in the category there is exactly one morphism (arrow) from it to o. In the category of sets, for example, singletons are terminal (since, from any set there is just the function identically the single member). Clearly, mappings from terminals are in one-one correspondence with elements of the codomain, set-theoretically speaking. Such mappings can do the work of elements, from a structural perspective.

[3] For a great deal of mathematics, weaker conditions suffice. Indeed, a topos based on a "local set theory" with just a natural number object as ground type—a so-called "free topos"—has been proposed as the natural categorical setting for constructive mathematics. (See [21], p. 206, and [4], p. 233.) Adding the law of excluded middle yields a theory determining "the free Boolean topos", thought of as the natural setting for ordinary classical mathematics. Determination, of course, is only up to equivalence, but even here there is a question of existence. The discussion below pertains to these proposals as well as to Mac Lane's and could just as well have been framed with respect to them.

to insure (more than) "enough sets" for ordinary mathematics, in the form of a model of (restricted) Zermelo set theory. ([27], 331 ff.) Thus, one might expect Mac Lane to take advantage of this and claim that the development of ordinary mathematics in category theory is unfamiliar only in its initial stages of recovering a powerful set theory, and that from there on it is plain sailing—or tedious copying and translating, depending on your point of view. Perhaps, however, Mac Lane has in mind an independent development of ordinary mathematics in topos theory which does not take a detour through set theory. An example might be the construction of a ring of "real numbers" in a topos of sheaves based on a topological space (as is done in [27], although even this mimics Dedekind's famous construction fairly closely). Then the claim may be a much more ambitious one, and would merit more attention than we can give it here. One would be showing that category theory is autonomous from set theory in a strong sense: not only is its primitive basis capable of standing on its own and sufficient for *some* recovery of ordinary mathematics, even if via a detour through set-theoretic constructions (call this "autonomy" *simpliciter*), but, without any such detour, it can achieve a genuinely distinctive, intelligible conceptual development throughout, not just in its initial stages. (Call this "strong autonomy".) This is, of course, not a precise distinction; perhaps we would only recognize "strong autonomy" if we saw it (perhaps some category theorists have already seen it?). But unless and until it is achieved, the charge that category theory is "parasitic" on set theory in its recovery of ordinary mathematics will surely linger.

This is a good place to recall an earlier debate between Mac Lane and Feferman on this topic. (See [9].) Responding to Mac Lane's claims on behalf of category theory as a possible foundation, Feferman argued, in essence, that category theory presupposes and uses, informally at least, notions of *collection* and *operation*, both in saying what a category is and in relating categories to one another through *homomorphisms* or *functors*. Moreover, it was argued, a foundational framework for mathematics must provide a systematic, theoretical understanding of these notions, something that set theory does, but category theory does not. It was explicitly recognized that alternatives to classical set theory (i.e. Z, ZF, NBG) might also provide this, so that the claim was not that category theory depends on established set theory *per se,* but rather that it is simply inadequate as a foundational scheme as it stands.

To my knowledge, Mac Lane never responded directly to Feferman's critique. But a response of sorts may be gleaned from subsequent publications, including those already cited above. There is frank acknowledgement that the notion of *function* is presupposed, at least informally, in

formulating category theory; indeed, category theory has been described as investigating the behavior of families of functions under the operation of *composition*. These are notions that the working mathematician understands and uses every day, and to build a theory on them is, in principle, no different from building a theory on the notion of *objects*, including *sets, belonging* to *sets*. Indeed, set theory itself (as axiomatized e.g. in the system NBG) can be recovered from a similar theory of *functions*, essentially through the identification of sets with their characteristic functions, as originally carried out by von Neumann [32]. So then we get two theories based on the notion of *function* making their respective foundational claims. Perhaps that isn't a bad thing; perhaps the notion lends itself to more than one systematic development. It then becomes a task for the philosopher or logician to compare them on fundamental issues.

Observe that this reply relies on an intended interpretation of '*composition*', as a binary operation on functions (which preserve various structural relations, depending on the context). When viewed this way, the reply seems to have some force. Every theory presupposes some informal concepts; one has to start somewhere. Why not with an informal notion of "collection of (structure-preserving) mappings related by composition" which one tries to systematize with axioms? Why, the category theorist may ask, do we need a *general* theory of collections and operations rather than merely a sufficient collection (a topos) of suitably interrelated maps in which ordinary mathematics can be carried out?

The main trouble with this as a reply to Feferman, however, is that it is diametrically at odds with the category theorist's *structuralism* vis-à-vis categories themselves. According to this, although the intuitive, everyday notion of *function,* with its many and varied instances preserving mathematical structure, indeed *motivates* category theory, this theory itself is presented *algebraically,* via first-order "axioms" only in the sense of *defining conditions* telling us what a *category* is, together with further ones defining *topoi* of various sorts. As such these "axioms" are like the conditions defining a group, a ring, a module, a field, etc. By themselves they *assert nothing. They merely tell us what it is for something to be a structure of a certain kind.*[4] As Awodey puts it,

"a category is *anything* satisfying these axioms. The objects

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[4]It is clear from [9], p. 150, that Feferman assumed this structuralist perspective on category theory:

"...*when explaining* the general notion of structure and of particular kinds of structures such as groups, rings, categories, etc. we implicitly *presume as understood* the ideas of *operation* and *collection*; e.g. we say that a group consists of a collection of objects together with a binary operation satisfying such and such conditions."

need not have 'elements', nor need the morphisms be 'functions', although this is the case in some motivating examples. But also, for example, associated with any formal system of logic is a category, the objects of which are formulas and the morphisms of which are deductions from premises. We do not really care what non-categorical properties the objects and morphisms of a given category may have..." ([1], 213.)

Indeed, this structural approach to category theory itself is of fundamental importance, not only for its claims of wide applicability—including self-applicability[5]—, but philosophically as well, as will be brought out further momentarily. Its immediate bearing on the above attempted reply to Feferman's critique is that it simply vitiates it. Just as group theory does not presuppose *product* in any ordinary sense as primitive, although ordinary usage does furnish motivating examples, neither can the category-theoretic structuralist fall back on a primitive, intended interpretation of 'composition', 'domain', 'codomain' in the way the set-theorist falls back on 'membership', viz. as providing an interpretation of the formal axioms according to which they assert truths (as the axioms of set theory are normally understood). In other words, category theory—at least, as presented in axioms—is *"formal"* or *"schematic"*: unlike the axioms of set theory, its axioms are not *assertory*.

Thus, a structural understanding of category theory actually underlies Feferman's critique: somehow we need to make sense of talk of *structures satisfying* the axioms of category theory, i.e *being categories* or *topoi*, in a general sense, and it is at this level that an appeal to 'collection' and 'operation' in *some* form seems unavoidable. Indeed, one can subsume both these notions under a logic or theory of *relations* (with collections as unary relations)*:* that is what is missing from category and topos theory, *both* as first-order theories *and, crucially, as informal mathematics*, but is provided by set theory. But it is also provided by second-order logic. This will serve us below, when we come to proposing a positive resolution for CT structuralism.[6]

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[5]Categories themselves with functors between them as morphisms are said to constitute a category, one of the thorniest aspects of category theory, giving rise to a version of Russell's paradox (if one is not careful), and to special efforts to put category theory on a sound set-theoretical foundation (see, e.g., [10], and the appendix by Kreisel).

[6]No doubt a logic or theory of relations of sorts can be recovered *internally* within a given topos, but what is required is an *external* logic of relations, enabling the introduction of categories and topoi in the first place. This involves *quantifying over relations,* as when a category is described as a collection of 'arrows' with a (some) relation meeting the conditions on 'composition', or when a theorem involving exis-

So far, we have seen that a structuralist understanding of category theory only highlights a dependence on a prior theory of relations. The situation regarding what we called "*strong* autonomy" is also problematic, apart from the Feferman critique. Consider, for example, the notion of a topological space. This is taken for granted in the ordinary workings of category theory, as the category of ("all") topological spaces with continuous maps between them as morphisms furnishes a prime example, and structures arising from topological spaces inspire the very concept of a topos. But the categorical foundationalist cannot take these notions for granted. The very notions of 'open set', 'collection of open sets closed under finite intersections and arbitrary union', 'inverse image of an open set', and 'continuous function' must be built up somehow from categorical primitives. Now it is true that a purely category-theoretic construction of key topological notions can be given in a topos using the notion of "internal locale", leading to a purely internal formulation of sheaf theory. (See, e.g. M&M, Ch. IX.) One may well ask, however, whether these developments would be (humanly) comprehensible without prior acquaintance with set-theoretic ideas. Granted, this moves beyond "autonomy" in a purely logical sense; but the goals of a foundational framework are surely epistemological as well as logical. We merely raise this issue here, without attempting seriously to address it.

Moving beyond the question of autonomy proper, we turn now to an equally important, intimately related problem, that of *mathematical existence.* This problem as it confronts category theory can be put very simply: the question really just does not seem to be addressed! (We might dub this the *problem of the "home address"*: *where do categories come from and where do they live?*) Of course, the first-order "axioms for categories" or those extended to include "axioms for topoi" do include existence claims, e.g. existence of enough morphisms e.g. for pullbacks or pushouts, existence of terminals, existence of morphisms for well-pointedness, existence of a "natural number object", existence guaranteeing the axiom of choice, etc. But, as already said, these axioms are to be read "structurally" *à la* algebra, that is, as *defining conditions*, not as absolute assertions of (putative) truths based on established mean-

tence of functors behaving in certain ways is proved, etc. Here again the disanalogy with set theory, as usually understood, is apparent. Precisely because the "axioms" for categories and topoi are to be understood structurally, we must appeal to some prior, external assumptions in order to prove any substantive theorems *about* these structures. What are these external assumptions? Set theory, with its assertory axioms, can stand on its own, and a logic of relations, indeed full blown model theory, can be carried out within the theory itself. Of course, one may not like *where* set theory stands, but an alternative foundational scheme must at least articulate alternative assumptions.

ings of primitive terms. Of course, such a use of the word "axiom" has a time-honored history, but it can obscure this fundamental distinction (which, we may remind the reader, lay at the root of the famous debate in correspondence between Frege and Hilbert). As already indicated, the axioms of set theory, as standardly read, are not "structural" in this sense (although it is an important issue in philosophy of mathematics whether it would be better somehow to read them structurally, as in [13], Ch. 2); rather, they are normally understood as specifying truths outright about "set-theoretic reality", "the cumulative hierarchy of (pure) sets". In this sense, set-theoretic axioms constitute a theory of mathematical existence, including the statement that there are infinite sets, that there are uncountably many uncountable cardinalities, that there are more ordinal levels than can be "measured" by any set, etc. And strong axioms of infinity, e.g. large cardinal axioms, claim directly that various abnormally large infinite sets exist as well.

Surely category theorists do not intend their first-order axioms for categories or topoi to be read in any such fashion. It is not as if the category theorist thinks there is a specific "real-world topos" that is being described by these axioms![7] The algebraico-structuralist perspective precludes this. But then, just as in the cases of more familiar algebraic theories, the question about mathematical existence can be put: *what categories or topoi exist?* Or, more formally, *what axioms govern the existence of categories or topoi?* On the assumption of autonomy, we're in the situation of the Walrus and the Carpenter, after the oysters were gone: "...but answer there came none..." [8]

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[7]Informally, of course, (s)he might have in mind sheaves of sets on some particular topological space, or some other mathematical construction. But then the question becomes, where do these spaces come from, or what axioms govern *their* existence. If set-theory is the official background, then at least we have an answer. But this violates autonomy.

[8]This is a bit of an exaggeration. A notable exception is [22], which presents first-order axioms, employing category-theoretic primitives, intended to capture intuitions about "the category of categories", including explicit existence axioms for certain elementary categories as well as a number of closure and "completeness" axioms. Space does not permit here a detailed assessment of this interesting effort. Doubts as to its consistency were raised early on ([17]), and, indeed, the stronger axioms of the theory are quite complex. Apparently, subsequent developments in topos theory overshadowed efforts to axiomatize a category of categories, but at the cost of suspending the articulation of existence axioms in the assertory sense, i.e. as credible truths outright rather than as merely part of an algebraico-structural definition. A detailed assessment of Lawvere's axioms would examine this issue of credibility, which in turn rests on some prior, not-merely-structural understanding of the primitives and intended interpretation, which does build in the notion of 'category' itself, and thereby presupposes the notion of 'collection' as well as that of 'functor'. (Cf. Bell [3].)

Now one might be tempted to appeal here to metalogic: assuming the first-order axioms for topos theory are formally consistent, doesn't the completeness theorem for first-order logic guarantee existence of a model, and isn't this sufficient for category theory? Even if we cannot construct a formal proof of consistency, due to Gödel's second incompleteness theorem, we could still regard the existence of a model as a working hypothesis on a par with formal consistency.

There are two main problems with this suggestion. The first is that the completeness theorem itself must be proved somewhere, and the usual "where" is of course set theory itself; model theory is the going framework for metalogic, and model theory is formally part of set theory.[9] Shouldn't the categorical foundationalist have to reconstruct metalogic categorically? If so, then how? One cannot simply appeal to the first-order topos "axioms" themselves, since they simply constitute a *definition* of a kind of category. Anything proved from such "axioms" only establishes the conclusion as holding "in any (or any possible) structure satisfying the axioms", which is a *generalized conditional* assertion, not a categorical one (in the ordinary sense of the word).[10] We are then in the old "if-then'ist" predicament that plagues deductivism: what we thought we were establishing as determinate truths turn out to be merely hypothetical, dependent on the mathematical existence of the very structures we thought we were investigating, and threatening to strip mathematics of any distinctive content (cf [29], 82 ff.).

The second problem is that models established by the completeness theorem may be non-standard, for example (by the downward Löwenheim-Skolem theorem) countable, and this is surely not what is intended in, say, the mathematics of continuous functions, not to mention general topology, which category theory surely wants to respect.

Now, from the perspective of the working mathematician, these matters may appear esoteric. The analyst, for example, often begins with the field and order axioms for the reals, and may view these structurally. It doesn't matter what satisfies them or where, ultimately, "the real numbers" come from. ("Somewhere we learned that such problems are all cleared up in set theory, which my colleagues accept as the going foundation...") One proceeds without addressing such questions and develops

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[9]Only a very weak set theory is needed to prove the completeness of first-order logic. In fact, a weak subsystem of second-order arithmetic (defined by the König infinity lemma) suffices. This doesn't affect the present discussion, as the axioms of that system are standardly treated as truths about numbers and sets thereof, not merely as a definition.

[10]Exploiting this ambiguity, one might well say that all categorical theorems (i.e. theorems of the structural axioms of category theory) are hypothetical, not categorical in the ordinary sense.

the subject. As pure mathematics, category theory can afford the same attitude, and that is precisely what we found above in M&M's "external point of view", category theory ultimately as part of set theory. But surely it is the job of a foundational framework to address precisely such problems. Then the category theorist must put on another hat; it is not enough simply to say, "We're not interested in these questions of mathematical existence". Such a stance may reflect a kind of *mathematical* virtue, in that it would normally be bad for mathematics to get too distracted by such questions; but it hardly qualifies as a *metamathematical* virtue!

Our conclusion thus far is that, as usually presented, category theory is defective as a framework for structuralism in at least two major, interrelated ways: *it lacks an external theory of relations, and it lacks substantive axioms of mathematical existence.* Somehow these need to be supplied if there is to be a genuine foundational alternative to set theory. Sections 5 and 6 of this paper will explore a novel way of doing this. But here one further puzzle concerning "mathematics in a well-pointed topos" is worth mentioning. As we have noted, Mac Lane only claims that topos theory suffices to recover "ordinary mathematics". In particular, it suffices for a version of Zermelo set theory. But what of Zermelo-Fraenkel set theory and beyond? And what of category theory itself? It does not seem true to the spirit of the subject to force all of it into a single topos, free or well-pointed or whatever. One can call all such mathematics "extraordinary", but shouldn't a structuralist approach to mathematics apply to these theories as well? From a purely mathematical point of view, the axiom of Replacement, for example, is very well motivated, even if it leads to infinite levels far beyond anything ordinarily needed or even contemplated. (We will return to this below.) And if recent work of Harvey Friedman on Boolean Relation Theory is a good indication, large cardinals of Mahlo type may come to play an essential role in answering a host of questions on the level of sets and functions of (n-tuples of) integers. How can topos theory accommodate strong axioms of infinity, and can it do so "autonomously"?

4   Bell's "Many Topoi" View Based on Local Set Theories

One of the most sytematic efforts to develop the view of category theory as a foundational alternative to set theory can be found in the writings of J. L. Bell (e.g. [4] and [5]). This is especially interesting from our own point of view as it expresses a "multi-universe" perspective remarkably similar in spirit to the modal-structuralism I have tried to develop.

Bell's proposal is based on a cluster of formal results in topos theory

to which he, Fourman [11] and others have contributed. The upshot of this work is that topoi, generally, can be realized, up to a precisely specified notion of categorical equivalence, as models of higher-order type theories based on intuitionistic logic. Indeed, one specifies a class of type-theoretic languages and theories (Bell's "local set theories") and shows that any topos has such a theory associated with it which in turns determines that topos up to categorical equivalence, a kind of functorial isomorphism ("up to isomorphism", one can say).[11]

Mathematically, this equivalence is useful in establishing general theorems for topoi, as one only needs to prove the theorem for all "linguistic topoi", the categories canonically associated with the local set theories. It also clarifies the sense in which topos theory can be seen as generalizing set theory: every topos is equivalent to one in which the objects can be thought of as sets and the arrows as functions between them satisfying the requisite closure conditions, with the proviso that the internal logic is in general intuitionistic rather than classical.

Although this representation theorem on topoi is very useful and helps the uninitiate gain access to topos theory, Bell emphasizes that that theory is far more general than set theory, admitting a wide class of realizations beyond ordinary set-theoretic hierarchies. Particularly important, for example, are *sheaves* of sets on a topological space, exploiting ideas of topology closely linked to the origins of topos theory.[12] In this setting, the subobject classifier generalizes the bivalent truth-value set of classical set theory, viz. as a Heyting lattice (in fact, of open sets of the underlying topological space), giving rise to internally definable logical operations obeying intuitionistic logic. Moreover, there arises a correspondence between familiar fixed or constant objects of classical set theory and *variable* objects in a topos of sheaves over a space X. For example, objects in the latter satisfying the definition of "Dedekind real

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[11]Categories **C** and **D** are *equivalent* if there is a functor F between them which is *full, faithful,* and *dense*, where $F:$ **C** $\rightarrow$ **D** is defined to be *full* when it carries the morphisms between any **C**-objects A and B onto those between $F(A)$ and $F(B)$ in **D**, *faithful* when F is one-one on those morphism classes, and *dense* when every **D**-object B is isomorphic in **D** to a value $F(A)$ of F, for some A in **C**. Further insight is given by defining a *skeleton* of a category as a maximal full subcategory with no non-identical isomorphic objects (where a subcategory is *full* just in case the arrows between its objects are just the arrows between them in the whole category), and then proving that two categories are equivalent just in case they have isomorphic skeletons. (The existence of skeletons depends on the axiom of choice.) Note that, in general, categorical equivalence abstracts from cardinality. (All this can be found in [4], Ch. 1.)

[12]Sheaves may be thought of as well-behaved function classes, closed under operations of restriction (e.g. to open subsets of open sets) and collation (piecing together when there is agreement on overlaps). For a full development, see [27].

number" correspond to real-valued continuous functions on X, viewed set-theoretically. Indeed, Bell argues, any topos can be thought of as a "local framework" for developing (ordinary) mathematics, and these multiple frameworks can be linked to one another by functors meeting special conditions, called *admissible* or *continuous transformations*. Shifting frameworks via such transformations reveals both a generality and a relativity of ordinary mathematical concepts, and what is typically thought of as absolute (in standard set theory) is thereby revealed to be only a special case.

This leads Bell to pursue an analogy with (special) relativity in physics: a particular topos corresponds to an inertial frame of reference, and a shift of frames is analogous to applying an admissible transformation of topoi. Certain concepts are recognized as frame-dependent, e.g. simultaneity or temporal order of space-like related events in physics, notions of set and number in the case of pure mathematics. Invariant or frame-independent relations are given prominence in special relativity; correspondingly, intuitionistically provable theorems of topos theory enjoy invariant status in mathematics, valid in any local framework. Rather than thinking of mathematics as ultimately set within a fixed, absolute universe of sets, it is seen as a combination of what is invariant over frameworks (topoi) together with that which depends on a choice of particular framework or frameworks suitable to contexts of interest. Illustrating this "many-worlds" perspective, the Axiom of Choice is seen as playing the role of *specification* to frameworks obeying classical logic,[13] rather than as an absolute proposition about the unique, real world of sets. Notably, the set-theoretic independence proofs of Cohen are incorporated by Bell into his framework in a straightforward way: Cohen extensions can be understood in terms of Boolean valued models of set theory, in which the standard set-theoretic universe is "expanded" by extending the two-valued set of truth-values to a larger Boolean algebra, and the models thereby generated can be recovered as topoi of sets "varying over" a suitably chosen partially ordered set P of items representing stages of knowledge (Cohen's forcing conditions). The resulting topoi share some but not all the structure of standard sets, depending on the choice of P, for example the Axiom of Constructibility may be forced to fail or the Continuum Hypothesis may be forced to fail, etc. On Bell's relativistic view, none of these topoi is privileged in fixing "absolute mathematical truth", so such principles are regarded as indeterminate,

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[13]As a clever construction of Diaconescu [8] shows, the Axiom of Choice for arbitrary domains implies the Law of Excluded Middle (in the presence of standard comprehension axioms). In fact, the mere assumption that the power set of a two-element set has a choice function suffices.

true in some topoi, false in others. Some mathematics, of course, turns out invariant over topoi in virtue of contructive provability, and may be regarded as absolute, but in general mathematics is relative to a topos as a background framework. This is Bell's "local mathematics".

From a structuralist perspective, this is clearly an interesting and attractive view. It provides a structuralist treatment of set theory itself; it avoids the troubling metaphysics of "the real world of sets"; it highlights interrelations among topoi as well as among the special structures of ordinary mathematics (taken as the objects of particular categories), often providing a deeper understanding of mathematical concepts, revealing new links via new kinds of generality, as illustrated by Bell's examples; and it shares other advantages associated with a pluralistic approach over a plurality of "absolutisms". Included among the latter is the view discussed in the previous section, mathematics in a (single) well-pointed topos, as well as the usual set-theoretic absolutism. What are the drawbacks?

It may appear, in light of some of Bell's own remarks, that the view goes too far in the direction of "relativism". In particular, Bell sees confirmation of the many-topoi view in the independence proofs of set theory, and one who doubts that the Continuum Hypothesis has a determinate truth-value (as Bell does) may be tempted to agree. On the other side, one who thinks that the question ("CH?") is perfectly well posed—even if difficult or perhaps humanly impossible to answer, and even if having an answer which is of little relevance to "ordinary mathematics"—might be tempted to take this determinateness as counting against the many-topoi view (agreeing on the claim of "confirmatory connection", this counting then as a case of "one person's *modus ponens* [being] another's *modus tollens*"). It seems to me that both are wrong. The very question, CH, is best understood as specifically addressed to a particular kind of structure, one containing (at least) the classical continuum and all subsets thereof (up to isomorphism, of course). Call such structures "sufficiently full" ("full" for short). Then, in Bell's terminology, the very question CH is already relativized to full frameworks, topoi of sets with a natural number object, all Dedekind reals constructed thereon, and the full power set of this (up to isomorphism). It would be better to write "CH_{full}", much as in special relativity one would write "$simultaneous_{rocket}$", to indicate relativization to "the rocket frame". Then obviously the fact that "CH_{full}" may have a different truth value from, say, "$CH_{constructible}$" or from "$CH_{Cohen extension(1963)}$" is perfectly compatible with both the determinateness of CH, in the usual sense, and the many-topoi view. Indeed, *exploiting* the analogy with special relativity, one should say that, once relativization to a topos or class

of topoi has been made explicit, one has restored frame-independence to the statement. (Whether events e_1 and e_2 are simultaneous$_{rocket}$ is frame-*in*dependent; the corresponding statement is true *simpliciter*, not merely "true in the rocket frame"!) The real "relativist" challenger to the determinateness of CH goes well beyond anything like the many-topoi view of mathematics and denies that it makes determinate sense to speak of "full structures" at all. It would be denied that we even unambiguously pick out an isomorphism class of topoi or classical structures with such language. This is prior to the many-topoi view and independent of it. Rather it limits what we can mean in trying to describe various topoi in the first place. We conclude that the interesting questions of determinateness of ZF-undecidables such as CH, projective determinacy, etc., are entirely detachable from the pluralism of the many-topoi view *per se*.

How does the many-topoi view fare with the problems raised in the last section confronting Mac Lane's suggestion (reconstructing (most) mathematics in a well-pointed topos)? Recall that we highlighted two such: the problem of autonomy from set theory, in particular the lack of a prior, external theory of relations, and the "problem of the home address", the mathematical existence of topoi. We also raised a third problem, related to the first, the problem of "extraordinary structures", containing e.g. large cardinals, or even just satisfying the Replacement axiom. In what sense can category theory accommodate these?

In connection with the first problem, we have already suggested that the many-topoi view does clarify the interrelations between category theory and set theory, spelling out just how the former generalizes upon the latter and also how local set theories provide for a canonical representation of topoi generally (via the equivalence theorem). The dependence on a prior theory of relations, however, emerges in Bell's work [4] at the level of choice of background metalogic in which local set theories—which are type theories with no cardinality restrictions on the symbol sets involved—and their models (categories and topoi) are described. At this level, Bell's presentation is, of course, informal, and, undoubtedly, various choices could be made for a background metalogic. It need not be full-blown set theory, perhaps, but some higher-order apparatus is required, as Feferman's critique would imply.

It may be, however, that the pluralistic view as described by Bell contributes something interesting on the question of "strong autonomy": the idea of "variable sets" and the interrelations among topoi expressed through admissible transformations among them are characteristically categorical in content, even if they could in principle be forced into set theory proper under some translation scheme.

When it comes to the second problem, the "home address" problem for topoi generally, the situation seems no better under the "many topoi" view than under the "arbitrary single (free, or well-pointed, etc.) topos" view. There simply is no theoretical account provided that addresses the question. Even the move from a given local set theory S to the category **C**(S) corresponding to S utilizes some unspecified comprehension principle for formation of equivalence classes of syntactic objects (the objects of the category, for instance, are taken to be equivalence classes of "set-like terms" of the language of S under the relation of provable equality). And construction of sheaf categories begins with a topological space X, just assumed given somehow. Furthermore, given some topoi, it is assumed that we may speak of suitably defined mappings between them (various functors) in a straightforward way, without worrying about the conditions under which such things "exist". Now it may be suggested that all such constructions can be carried out internally, within a given topos with a natural number object. Still, we can ask about the source of such topoi, as well as the conditions under which any given one can be transcended. Of course, if one is just doing mathematical constructions, it is standard practice not to specify such things as the justification for forming equivalence classes or the source of a topological space, of functors, or even of topoi themselves; but it is also standard practice to fall back on set theory if questioned about such matters! And, as we have been emphsizing throughout, that is precisely what is not allowable in the context of CT structuralism. (Nor should it be in "toposophy", as CT enthusiasts have called foundations of mathematics based on topos theory).

There is an interesting hint by Bell that at a fundamental level modal logic might have a role to play. At the very end of his [5] he writes:

> "...the topos-theoretical viewpoint suggests that the absolute universe of sets be replaced by a plurality of 'toposes of discourse', each of which may be regarded as a possible 'world' in which mathematical activity may (figuratively) take place. The mathematical activity that takes place within such 'worlds' is codified within local set theories; it seems appropriate, therefore, to call this codification *local mathematics...*" (245)

We will return to follow up on this hint in the next two sections.

Finally, in connection with the third problem raised above on accommodating large set-theoretic structures, Bell cites [28] where the basic equiconsistency link between topos theory and (restricted) Zermelo set theory (with bounded separation) is extended to ZF set theory and beyond. The way in which this is achieved, however, is by a kind of "brute

force". In the case of the Axiom of Replacement, one simply rewrites it in category-theoretic notation and adds it as a constraint on topoi.[14] But this shows little more than that the language of topos theory is rich enough to *express* the axiom. What it fails to provide is any independent, category-theoretic reason for "believing" the axiom—better believing in the existence of structures satisfying it—or even its consistency, or any new motivation for adopting it. In the case of Zermelo set theory, CT does provide this, at least by pointing to all the situations in mathematics which fit naturally within Cartesian closed categories, and it is these conditions, generated from within category theory, that lead directly to modelling of (restricted) Zermelo set theory. So far as I can tell, nothing like this is provided for the Axiom of Replacement. In one way or another, it is added "by hand". The motivation for the axiom remains characteristically set-theoretic (to guarantee that the domain of sets and functions is incomparably more vast than any of those objects, that it extends far enough so that transfinite recursion will be legitimate, so that ordinal arithmetic operations will be well-defined, etc.). In this regard, it seems hard to deny that CT is still parasitic on set theory.

5   Category Theory and Modal Structuralism

As we have argued, category theory as it stands does not adequately address questions of mathematical existence and so has difficulty competing with set theory on this score. But, as already acknowledged, set theory has problems of its own, some of which are engendered by the very commitment to a single, maximal universe of mathematical objects, something category theory seems to avoid. Indeed, this is one of its attractions, and, as we have just seen, it is a cornerstone of Bell's "many-topoi" proposal. In a similar spirit, Mac Lane has written:

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[14]It may be claimed, however, that [20] improves on this in its systematic, interesting development of "Zermelo-Fraenkel Algebras" in a category-theoretic setting (the background structure is a Heyting pre-topos). The starting point is a set of axioms, in the language of basic category theory, defining a notion of "small map". The construction of Zermelo-Fraenkel algebras and the recovery of ordinal number structures depend, however, on a condition (on "small maps") called "Collection", which is a category-theoretic generalization of the statement (in a set-theoretic setting) that a cover of a small collection (i.e. set) has a small subcover, which is a version of the "Collection Principle" of set theory, a well-known classical equivalent of Replacement. Cf. [18], pp. 72-73. (Intuitionistically, Collection is actually somewhat stronger, a result of Friedman [12]. Cf. [2], 163.) While the Collection requirement on "small maps" seems natural enough if one is aiming at a "large domain" in comparison with any of its elements, the very same can already be said of the Collection Principle of set theory. An even simpler and more direct expression of this idea, leading to Replacement, will be given in the language of mereology and plurals in the next section.

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> Understanding Mathematical operations leads repeatedly to the formation of totalities: the collection of all prime numbers, the set of all points on an ellipse, the manifold of all lines in 3-space...the set of all subsets of a set..., or the category *Cat* of all topological spaces. There are no upper limits; it is useful to consider the "universe" of all sets (as a class) or the category *Cat* of all small categories as well as CAT, the category of all big categories. This is the idea of a *totality,* and these are some of its many formulation. After each careful delimitation, bigger totalities appear. No set theory and no category theory can encompass them all—and they are needed to grasp what Mathematics does. ([26], 390)

And on the level of formal systems:

> We cannot realistically constrain Mathematics to be a single formal system; instead we view Mathematics as an elaborate tightly connected network of formal systems, axiom systems, rules, and connections. ([26], 417)

These are clear acknowledgements of the futility of seeking to embrace all of mathematics in a single framework, formally or ontologically. Surely, Mac Lane's foundational claims on behalf of topos theory must be understood in their light: whatever the representational powers of a well-pointed topos, they are ultimately limited and readily transcended. There is an open-endedness, incompleteability, or indefinite extendability that is an essential aspect of mathematics. Similar views have been expressed by eminent set theorists as well, such as Zermelo (1930), and they form a cornerstone of the modal-structuralism I have tried to develop. ([13], [14]) Especially in light of Bell's many-topoi view just reviewed, all this suggests that category-theoretic structuralism ("CTS") should somehow be combined with modal-structuralism.[15]

According to this view, talk of mathematical objects generally is understood in the context of entertaining logico-mathematical possibilities, typically expressible in a version of S-5 second-order modal logic with extensional comprehension. (See [13] and [14].) As will be explained

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[15]One might also consider the *ante rem* structuralism of Shapiro or Resnik (what I called *sui generis* structuralism in [15]) as a home for category theory and its structuralism. That is beyond the scope of this paper. We would only point out here that the kind of open-endedness and extendability that Mac Lane recognizes is *prima facie* in conflict with a fixed universe of structures and positions in structures to which the *sui generis* approach seems committed. This is an analogue of set-theoretic structuralism's problem of "the unique cumulative hierarchy of sets".

further below, one countenances classes and relations (better, nominalistically acceptable substitutes of these) *within* a possible world, as it were, but not "across worlds". Talk of possible worlds is heuristic only; officially one only entertains that certain things are possible, and mathematical demonstrations are understood as telling us what would necessarily be the case were the relevant structural conditions fulfilled. The problem of the "home address" is "solved" by elimination: officially there need be no actual commitment to objects at all, only to what (propositionally) might be the case. The usual apparatus of singular (and plural) reference and quantification over objects is recovered, but only as devices for describing and reasoning about what might or must be the case. The possibility commitments are explicitly spelled out in axioms of modal-existence, guided by mathematical practice, i.e. by the characterizations of structures of mathematical interest already provided by mathematical and logical work. Typically, any reasons Platonists can give for believing their absolute existence claims can be adapted as reasons *a fortiori* for the modal-existence postulates. Although the core modal logic is held constant, these modal-existence postulates must be added to obtain modal-structural analogues of specific mathematical theories, and there is no pretense of a single, all-embracing system. Formally, the open-endedness suggested by Mac Lane's remarks is respected. Ontologically, it is as well, for the restricted modal comprehension scheme does not permit commitment to any maximally rich possibilities. One cannot form the "union" of "all possible mathematical universes". One may even explicitly add an *Extendability Principle,* to the effect that any "world" has a proper extension.

At first blush, one might suppose that to achieve a synthesis of Bell's "many-topoi" view and modal-structuralism, there is nothing to do. The topoi are, up to equivalence, specified by theories, and each of these can be taken as specifying a logico-mathematical possibility, so that Bell's view simply results from applying the modal-structural scheme to topos theory in the most straightforward way. On closer analysis, however, matters are not nearly so straightforward.

There are two main obstacles to such a modal-structuralist interpretation of topos theory. First, the local set theories Bell describes are set in languages with no cardinality restrictions on the symbol sets (type symbols, function symbols). Most of these theories cannot be written down. The only way, then, to assert the possibility of structures for them or to assert hypothetical generalizations about what must hold in such structures is to introduce a general *satisfaction* relation between structures and theories. Thus, the straightforward, intrinsic, second-order expressions of these things that are available for modal-structural

interpretations of number theory, analysis, and set theory—employing finitely many axioms, substituting relation variables for constants and relativizing to a domain, as in [13]—are unavailable here. But how is *satisfaction* to be introduced without falling back on a prior set theory, which, in the interests of autonomy, category theory seeks to avoid? In other words, Bell's program is set within a metalogic of local set theories, and if we "begin" with it, we are implicitly beginning with a framework rich enough to carry it out. That framework cannot be category theory itself, since category theory is being introduced by appeal to local set theories. It seems that a rich set theory must be presupposed.

The second problem is related to this. How are we to make modal-logical sense of functorial relations among many categories, which is of the essence of category theory? Asserting the possibility of this or that particular (type of) category or topos, one at a time, will not do, for we need to consider functional relations between and among different ones, and if they are not entertained as coexisting, the idea of a functor "between them" literally makes no sense. This is just a special case of modal-structuralism's avoidance of "transworld relations", a consequence of avoidance of literal quantification over worlds or "possibilia". I cannot quantify over relations between actually existing things and "things" that merely might have existed, for there literally are no such things. Similarly, I cannot entertain the possibility of relations among some things that might have existed and some others that then still would not have existed but merely might have. What I can do is entertain further compossibility assumptions, that the items between which I want to entertain relations might have coexisted. (This amounts to applying second-order comprehension "within a world".) In the abstract setting of pure mathematics, such compossibility assumptions seem perfectly reasonable, but the trouble is that we have to assert them whenever we wish to compare two distinctly hypothesized structures. This works so long as we only care about comparing finitely many structures at once, but category theory is distinctive in systematically interrelating many categories at once (not merely the structures as "objects" within a given category). And here "many" means "indefinitely many". No wonder category theory usually proceeds by simply presupposing some large background universe of sets!

It seems, then, that, in order to do justice to category theory, enough objects must be "simultaneously" available to support many categories, topoi, and functors between them. Does this mean that CT is ultimately dependent on set theory after all? Surprisingly, the answer is no. As it turns out, the conceptual resources for positing sufficiently rich domains—and they need be posited only as logical possibilities, not as

actually existing—are available even to a nominalist, employing mereology and plural quantification; set-theoretic notions of membership, class, singleton, etc. are dispensable. Put in other words, the combination of mereology and plural quantification already incorporates just enough of the content of set theory to do the job. In effect, this combination gives the expressive power of full second-order logic, once the possibility of infinitely many individuals is postulated (as it must be in any case for CT), and this suffices to express conditions guaranteeing even inaccessibly many objects (in the sense of strongly inaccessible cardinals). The proposal then is that CT and topos theory can be carried out relative to any such background domain or universe. It need not be a universe of sets; it need not be a privileged topos. Indeed, the objects are left entirely unspecified, and there is no mathematical structure presupposed beyond that implicit in mereology and plural quantification, i.e. arbitrary wholes of any individuals exist and they may be quantified over in the plural as well as singular sense.[16] Beyond this, the sort of background postulated is mathematically neutral; but it is rich enough to support CT as well as set theory and indeed any of the structures of ordinary mathematics. Finally, any such (possible) domain can be properly extended, so that there is an unlimited, open-ended range of ever more encompassing, indeed incomparably larger, universes relative to which CT or set theory can be developed. The result is a kind of double relativity: there is the relativity that Bell highlights of particular mathematical theories and concepts to different topoi, and then there is the further relativity to background universe supporting topos theory itself. How is all this possible on so meagre a basis? Here is a sketch.[17]

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[16]For motivation and details on plural quantification, see [6], [24], and [33].

[17]Much of what follows is inspired by Lewis [24]. The main differences are these: (1) Lewis takes himself to be describing "Reality", whereas we are defining the notion of "large domain for mathematics", and would only postulate such as a logical possibility. (2) Lewis is engaged in a reconstruction of set or class theory, and takes the set-theoretic notion of "singleton" as primitive, thereby combining in a highly unorthodox way set theory and mereology. (In Lewis [25], however, "singleton functions" are defined in mereology plus plural quantification and are proved to exist; nevertheless, the notion of 'class' is taken as absolute in the background.) We dispense with set-theoretic primitives entirely, and the notion of 'class' in an absolute sense as well, and would read Lewis [25] as providing a relative interpretation of set theory inside large universes in our sense. (3) Some of the definitions ("small", "few", etc.) of Lewis [24] were more complicated than necessary for lack of the means of ordered pairing, developed by Burgess, Hazen, and Lewis in the Appendix to that work [7]. The definitions given below, independently arrived at, rely heavily on such ordered pairing; some of these definitions, I later learned, Lewis had already given in his [25]. (4) The version of Replacement we recover below is more general than Lewis' and is derived without special extra assumptions (cf. [25], 22).

6   Theory of Large Domains

First, we will take our background logic to be classical,[18] monadic second-order quantified S-5 modal, without the Barcan formula, with restricted extensional comprehension, applied to formulas built up from the primitive binary relation, $<$, for 'part of', i.e., the necessitation of

$$\exists x(\Phi(x)) \rightarrow \exists X \forall y[Xy \leftrightarrow \Phi],$$

where Φ is modal-free (and lacks free 'X').[19]

In addition, we adopt a "comprehension" scheme for individuals, saying that the whole (or "sum" or "fusion") of any individuals satisfying a given condition exists. First define $x \circ y$, "x overlaps y", as $\exists z(z < x$ & $z < y)$. The scheme is then the necessitation of

$$\exists x \varphi(x) \rightarrow \exists u \forall y[y \circ u \leftrightarrow \exists z(\varphi(z) \ \& \ z \circ y)],$$

where, again, φ is modal-free (lacking free 'u'), but may contain monadic second-order variables, free or bound. (The antecedent is to avoid commitment to a null individual.) Occasionally, we shall invoke a suitable version of the Axiom of Choice. (And why not? Thinking classically about multiplicities of objects, it is obviously true!)[20] For most of what follows, modality plays no role. It enters only in asserting the possibility of large and ever larger background domains.

Let us take as the goal to define a plurality X of atomic individuals to have strongly inaccessible cardinality,[21] where an *atom* a is an individual with no proper parts (written $\forall b(b < a \rightarrow b = a)$), and an individual is *atomic* just in case it is composed entirely of atoms (also expressed, "is a sum of atoms" or "is a fusion of atoms"). (For simplicity, we can ignore non-atomic individuals ("gunk"), but this turns out to be no real limitation.) We use upper case letters and second-order logical notation to indicate pluralities, e.g. $\exists X(Xx)$ is read "there are some things one of which is x"; and (when $\in$ is available) $\exists X \forall y(Xy \leftrightarrow y \notin y)$ says, harmlessly, that there are the non-self-membered things. (Thus predication

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[18]This is our "external" logic; choosing it to be classical is fully compatible with the fact that the internal logic constructed within topos theory is intuitionistic.

[19]The antecedent non-emptiness condition accords with the plural reading of second-order quantifiers: "there are exactly the $\Phi's$" should fail if there are none.

Note that the universal quantifier is not boxed; in the idiom of possible worlds, only those objects occurring "at a world" are quantified plurally ("form pluralities", one may say, without implying commitment to a new kind of object, "pluralities"). *Possibilia* are not recognized.

[20]The means of expressing choice principles, i.e. quantifying over functions and relations, will be sketched momentarily.

[21]To be sure, for many purposes topos theory can survive on much less; but inaccessible levels arise naturally in describing set-theoretic structures, which topos theory aims to capture (in pursuit of "toposophy"). They are also useful in making sense of the distinction between 'small' and 'large' categories.

does not mask a relation such as membership between objects; it is truly an ontologically innocuous classification device. Upper case variables do not range over special objects ("pluralities" mistakenly reified) but just enable us to keep track of which many objects we're saying what about (which sounds a bit too much like Humpty-Dumpty but should be clear nevertheless).)

The first condition to impose on atoms X is that they be infinite. This is easy. Call some individuals Y *nested* just in case each one of them is a proper part of some other of them. Then define X to be *infinite* just in case there are some $Y,$ parts of (the fusion of) $X,$ which are nested. (Note that, whereas X may be taken indifferently as a plurality or as a fusion of atoms, as decomposition into atoms is unique, Y is essentially plural in this definition, as it is necessary to individuate some non-atomic individuals as nested. Plural talk builds in principles of individuation just as class talk does, making up for a major defect in talk of wholes.) We require

(1) X is infinite.

The next step is to appeal to one of the clever combinations of mereology and plural quantification worked out by Burgess, Hazen, and Lewis [7] that gets the effect of *ordered pairing* of arbitrary individuals. This rests on the assumption that infinitely many atoms are available. In Burgess' version, easiest to describe to mathematicians, one speaks of unordered pairs ("diatoms", fusions of two atoms) coding two one-one correspondences between all the atoms (in our setting, all the atoms of X) and two respective disjoint proper parts of them (their totality). (A harmless assumption, "*Trisection*", of partitionability of the the totality of atoms of X into *three* pieces, with diatoms coding a one-one correspondence between any two pieces and part of the remaining one, enables thorough disjointness of the ranges of two total one-one correspondences.) One keeps track, by order of quantifiers, of a first of these one-one correspondences and a second, and codes ordered pairs, first of atoms a and b, as the sum of the first image of a and the second image of b. The first (second)-image of an arbitrary individual is the sum of the first (second)-images of its atoms, and then one defines the ordered pair of two arbitrary individuals as the sum of the first-image of the first plus the second of the second. (All this is relative to a given trisection of all the atoms, which involves an ordered string of plural quantifiers.) Armed with all this, one then can get the effect of quantification over arbitrary n-ary relations of individuals via plural (monadic) quantification over pairs, etc. Mereology thus enables a reduction of polyadic to monadic second-order quantification.

With this much it is now fairly straightforward to impose analogues

of the axioms of Power Sets and Replacement on "small parts" of X, guaranteeing that X will be inaccessibly large. It is in fact possible to do this in polyadic second-order logic without invoking mereology, but we needed mereology anyway to get infinity and ordered pairing, and retaining it facilitates the constructions for the analogues of Power sets and Replacement. First, call an individual p *part of X* if it is part of the fusion of the X, and call a part p of X *small (relative to X)* just in case there is no one-one correspondence between the atoms of p and those of X. (If there is such a one-one correspondence, p is *large (relative to X).*) Now (1) should be strengthened to

(1′) *Some small part of X is infinite.*[22]

Now, for the effect of Power Sets, we can require

(2) *For any small part p of X, there is a q, part of X, such that q is small and there is a one-one correspondence between the parts of p and the atoms of q.* (That is, there are as many atoms as there are parts (arbitrary sums of atoms) of p, and X is so large that the sum q of those atoms is still small. Since the parts of p are in one-one correspondence with the non-empty subsets of p's atoms, this is a mereological-plural analogue of the Power Set axiom of set theory. q can be called the "power object" of p (relative to the assumed one-one correspondence). Strikingly, the notions of "set" or "membership" are not really needed.)

To get the effect of Replacement, we can seek to derive:

(3) *Let p be a small part of X and let the R be ordered pairs coding a functional relation between the atoms of p and some small parts A of X; then the fusion of the A is also small.* (A serves as the range of R as a function, set-theoretically speaking.)

In fact, this can be derived from an assumption about "unions" (cf.[25], "Hypothesis U"). Call a plurality A of parts of X *few* if there is a one-one correspondence between them and *some but not all* the atoms of X. Then the assumption is

(3′) *The fusion of few small things (i.e. parts of X) is small.*[23]

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[22] In order to facilitate the recovery of natural number structures, one can complicate the definition of 'the Y are nested' slightly:

"One of the Y is an atom, a, and for each x of Y there exists a unique atom b not part of x such that the fusion $x+b$ is one of them."

A smallest plurality of atoms of the form

$a,\ b,\ b',\ b'',\ ...,$

generated from the assumption of Y in the obvious way, will then serve as domain of a natural-number structure. ("Smallest" here is in the sense of inclusion; an unabashed appeal to impredicative second-order logical comprehension is made, but predicativity is not one of our aims here.)

[23] Assuming a choice principle implying that the atoms of small things can be well-ordered, (3) follows from (3′), as the A are few small things. (To prove that the A are few, map each y of the A to the least atom x of p such that $R(x,y)$.) The

Note that we have the resources of full second-order logic, so that this expresses in effect the second order *axiom* of Replacement, not merely the first-order axiom scheme. Proving that X has strongly inaccessibly many atoms is facilitated by the Axiom of Choice. One can introduce analogues of ordinals and then give a relative interpretation of ZFC^2 inside X.[24] The strongly inaccessible cardinality of X then follows by standard results going back to Zermelo (1930).

Of course, one needn't stop here. One can carry out the above constructions relative to X which is then stipulated to be *small relative to X'*, a more encompassing totality. Since any such X can be identified with the whole of its atoms, one can further require that there be some inaccessibly large wholes together with a one-one correspondence between them and the atoms of one of them, guaranteeing inaccessibly many inaccessibles inside X'. Indeed, a part of X' can be required to be hyperinaccessible by requiring a one-one correspondence between (the plurality of) its parts of inaccessible size and its atoms. Such a part can also be required to be small relative to X', and so on. (Or X' can be considered a *small part of X''*, etc., without conceivable end.)

Our suggestion then is that category theory and topos theory can be understood as carried out relative to a large background domain X described along these lines, postulated as a logico-mathematical possibility. It should be clear from what has been presented that many standard examples of categories and topoi can be explicitly constructed within such X, or possibly an extension of X, using the mereological-plural machinery (including the above stated comprehension principles).[25] Indeed, even a general notion of *satisfaction by structures*, on which we saw category theory tacitly dependent (above, p. 7), can be reconstructed with

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other direction is also evident, given the positive part of the definition of '*few*': let A be few small things and let p be the fusion of the atoms of the range of the given one-one correspondence (call it R) from the A. Clearly, p must be small, and then, by Replacement (3) governing the inverse of R, the fusion of the A is small, qed.

This improves on Lewis' argument in [25] in two respects: (3) is more general than the version of Replacement given in [25] (p. 22) in applying to functions from atoms to small things, not just to atoms; and, as just argued, (3)—and *a fortiori* Lewis' version— need not be assumed as an extra principle of megethology but already follows from Lewis' "Hypothesis U" (which is (3′)) and Choice, which Lewis also assumes.

[24]Lewis (1993) in effect contains a proof of this, via introduction of a "singleton function", from which a suitable "membership" relation can be recovered. ($x \in y$ iff the singleton of x (an atom) is part of y).

[25]A brute force demonstration of this can be given by first developing an interpretation of ZF inside such an X; then any categories definable from sets in the usual way will exist inside X. But typically it is not necessary to resort to this as a direct construction can be given instead.

this machinery. (Of course, such a notion would be relative to a given, sufficiently large background domain.)

To illustrate the point about explicit constructions, consider the category of small groups in X. (Let us write "small$_X$" to indicate "small relative to X".) As will emerge, the objects of any small$_X$ group can be taken as atoms (recall that the parts of any small$_X$ individual correspond one-one with some atoms), but what of the group binary operation, "multiplication"? It can be understood as a plurality of ordered triples (relative to the apparatus of pairing described above). But to form the category in question we need to "collect" groups, *with* their operations somehow, and homomorphisms between them as the arrows of the category. How can sense be made of this if we only describe the group multiplication plurally? The answer is that we have a method of passing from any plurally given group operation on a small$_X$ group to a *bona fide* individual of X. Each ordered triple of the operation is an individual; as part of the whole of all such triples, which is small$_X$, it corresponds to an atom of X (by the construction of X). The sum of all such atoms then codes the group operation, and it is small$_X$. By a further one-one correspondence, the whole operation is even coded by an atom; moreover, a small$_X$ group, as a pair of its domain and its multiplication (from which the unary inverse can be derived) can itself be coded as an atom. *Thus, category theory's treatment of structures as "pointlike" is literally realized in this representation, even while allowing—like set theory—for as much internal analysis of structures as one likes!* In this way any small$_X$ operation or function is representable as a small part of X, and so can be collected or operated on by other functions. (Thus, in particular, the transformations of small$_X$ transformation groups can be coded as atoms.) Homomorphisms between small$_X$ groups can be treated in the same way. Thus the category of such groups can be introduced, via plural constructions, over X. Analogously, functors between small$_X$ categories exist as small$_X$ objects in X. However, the category of small$_X$ groups is itself not small$_X$. (It is only "locally small", i.e. the (sum of codes of) homomorphisms between any two small$_X$ groups is itself small$_X$.) Inside X we have no way of recognizing it or operating on it functorially except via plural quantification. We can speak of "the small groups of X" but we don't have any small part of X corresonding to this totality. For certain purposes, this is harmless, but for others, e.g. forming functor categories over X, we need objectual representations that are "collectible", retaining distinctness from other objects of the same sort. Without codability by atoms, we lack the means to insure this inside X. Here is where appeal to larger universes enters. We may consider any proper extension X' of X such that X is small$_{X'}$. Then the

category of small$_X$ groups is small$_{X'}$, and *there* is codable as an object operable by functors which will also be codable.

A similar situation obtains for topoi. It should be clear that many topoi can be introduced into large backgrounds X as we have described them; the additional structure of binary products, a terminal object, subobject classifier, and power objects need not take us beyond totalities still small relative to X. Indeed, the main worry would be over power objects, but our construction of large X is almost tailor-made for these, at least insofar as cardinality is concerned. It is only if a CT version of Replacement is imposed that a topos would have to be too large to be small$_X$. But even then, in order to be available as the domain of functors, such a topos can be embedded in a proper larger extension X' of X where it will be small$_{X'}$.

It should also be clear that the Equivalence Theorem highlighted by Bell can be proved relative to any given large background X. Local set theories themselves as syntactic objects can be treated mathematically, as is called for when the symbol sets are uncountably large. Given that set theory itself is interpretable in X, the semantics of local set theories with topoi as models should also be. Of course, on such a reconstruction, the Equivalence Theorem can only speak of topoi inside X; to transcend this, one must step up to a larger extension X', but then one will confront an analogous limitation. But, it would be our contention, that is as it should be. A theorem such as the Equivalence Theorem, which purports to govern "all possible topoi" can do so but only in the sense of governing any topos living in a large domain $X;$ that is, *any* possible topos is equivalent to some canonical one, model of a local set theory, in the same domain; but that does not mean that all possible topoi can be found "all at once" living in a single domain. In this respect, topos theory and set theory are on a par. Just as any universe of sets can, logically speaking, be properly extended, so can any plurality of topoi. Still that does not prevent us from proving things about "all possible sets, ordinals, cardinals, etc.", so long as we're careful in our interpretation of such results, which relativizes every proper mathematical result to a given domain.

In general, structures large in X become small in proper extensions of X. This is true of categories as well as sets. A corollary of this way of understanding category theory is that the "small/large" distinction is not absolute but relative to a background universe of discourse. This may already be implicit in treatments which take an ambient set-theoretic universe as background where one does not insist that it be "all sets"; but here we make the relativity explicit. Thus, our understanding of "large categories" is closely analogous to Zermelo's understanding (1930) of

proper classes. Such things are relative to a given domain of sets; classes proper relative to $\mathcal{D}$ become sets in proper extensions $\mathcal{D}'$ of $\mathcal{D}$ (which, by Zermelo's construction, have larger inaccessible height than that of $\mathcal{D}$). In the present setting, the same situation obtains: categories large relative to X become small in proper extensions X'. The additional effect is that, as small in X', they become codable as first-order objects as well, potential arguments of further operations and items of further structures. Echoing Mac Lane, there are no limits, only relative stopping points (echoing Zermelo [34], 47). There is also the difference that modality makes: we are not even tempted (or shouldn't be, once we reflect) to speak of a totality of "everything there might possibly be". . (For those who still yearn for the absolute in the realm of the infinite, we would simply say: You can't have *everything*!)

7 Summary and Conclusion

As its proponents have maintained, category theory does offer an interesting structuralist perspective on most mathematics as we know it. But it needs to be supplemented and set within a yet wider framework. As explained above, category theory—with its non-assertory, algebraico-structural axioms—depends on a prior notions of *structure* (collection with relations) and *satisfaction by structures* to make sense of the very notions of 'category' and 'topos'. On the ontological side, we have argued that category theory is insufficiently articulate. To avoid a collapse to "if-then'ism", it requires substantive axioms of mathematical existence, at least of background universes as logico-mathematical possibilities. Although this usually manifests itself through an introductory appeal to an unspecified, ambient domain of *sets,* this is not necessary. Instead, one can develop a modalized *theory of large domains* relying on the more neutral and general or schematic notions of "part/whole" and plural quantification, as described above. Relative to such domains, both category theory and set theory can be developed side-by-side, with neither viewed as prior to the other. Instead, each makes its own contribution and complements the other. Moreover, the problems and puzzles engendered by "large categories" and "proper classes" are solved in a uniform way, through the indefinite extendability of large domains, realizing ideas of both Grothendieck, for category theory, and Zermelo, for set theory.

Our proposal thus incorporates the relativity that Bell has highlighted of ordinary mathematics to a given topos as background. But it brings out a further relativity of topoi to background domains, resulting in an overall double relativity. This in no way threatens the validity or objectivity or even the generality of mathematical results. Rather it

enables us to understand them in a way that respects the open-ended, incompleteable character of mathematics. At the same time, while accommodating the structures defined by both set theory and topos theory, it removes any dependence on actual existence, as it is only possibilities that matter for pure mathematics; and it even reconstructs a rich theory of relations, via the language and logic of plurals, without reifying them—except via coding when we need to! We can have our cake—when doing one of philosophy or mathematics—and eat it—when doing the other. Even proverbs have a relative interpretation.

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