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Mathematical Constructivism in Spacetime

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To what extent can constructive mathematics based on intuitionistic logic recover the mathematics needed for spacetime physics? Certain aspects of this important question are examined, both technical and philosophical. On the technical side, order, connectivity, and extremization properties of the continuum are reviewed, and attention is called to certain striking results concerning causal structure in General Relativity Theory, in particular the singularity theorems of Hawking and Penrose. As they stand, these results appear to elude constructivization. On the philosophical side, it is argued that any mentalist-based radical constructivism suffers from a kind of neo-Kantian apriorism. It would be at best a lucky accident if objective spacetime structure mirrored mentalist mathematics. The latter would seem implicitly committed to a Leibnizian relationist view of spacetime, but it is doubtful if implementation of such a view would overcome the objection. As a result, an anti-realist view of physics seems forced on the radical constructivist.

1 The main questions

Constructivist mathematics based on intuitionistic logic is typically motivated by a focus on the computational capacities of agents, of human minds or of idealized mathematical inquirers of some kind. Indeed, mathematical statements themselves are understood as expressing computational content: in contrast to the classical conception of mathematical propositions with determinate truth status, about an objective, abstract subject matter consisting of numbers, functions, sets, or other objects or structures, constructivist propositions are associated with *proof conditions*, forming the basis of a distinctive interpretation of logical connectives. 'The law of the excluded middle', for example, expresses that any proposition either is provable (in some absolute sense) or is refutable, something that no one should wish to invoke as part of one's *logic*. And, as is well known, *existence* is interpreted constructively: to prove that an object o satisfying a condition C *exists* is to have a method of exhibiting or constructing o together with a constructive proof that o satisfies C. And, strikingly, a universal claim of the form $\forall x C(x)$ carries with it a strong *existential* import to the effect that some method is available for proving, of any x in the domain of quantification, that it satisfies C. On a thoroughgoing intuitionistic interpretation of mathematical statements, according to the standard way of explaining the meanings of the intuitionistic logical operations, mathematics turns out literally to be about the computational and demonstrative capacities of certain idealized agents, or perhaps about a special realm of 'constructions' and 'proofs' discoverable by such agents. In any case, it is not tailored to an objective interpretation along classical set-theoretic or structuralist lines.[1]

For the most part, constructive mathematics in the Brouwerian tradition has been pursued as pure mathematics, with little concern for applications in the sciences (with the obvious exception of 'computer science', at least on the part of constructivists accepting Church's Thesis). Yet, as Weyl proclaimed, 'it is the function of mathematics to be at the service of the natural sciences' (Weyl [1949], p. 61). Indeed, fulfilling this function stands as a necessary goal of any proffered substitute for classical mathematics. An alternative that falls short in this regard is surely not viable as a *substitute*, notwithstanding the strengths and advantages it may offer as a *companion*.

Since the seminal work of Errett Bishop [1967], it has been clear that, in fact, constructive mathematics is surprisingly rich in its capacity to develop serviceable constructive versions of classical mathematical theories central in scientific applications, including the theory of metric spaces, Banach and Hilbert spaces, and a good theory of measure. There is little doubt that most ordinary scientific applications of mathematics can be fitted within Bishop's constructivist framework (which employs intuitionistic logic but avoids the Brouwerian theory of choice sequences). However, 'most' is not 'all', and not all important scientific applications are 'ordinary'. As Beeson notes in his survey, the calculus of variations, for example, especially in its concern with *existence* of extremizing functions, 'lies right on the frontier between

[1] Our focus in this paper is on a radical constructivist view as just described, which treats mathematics as having a special universe of constructions or constructive objects as its subject matter, together with a logical apparatus understood in terms of computational capacities to build up or 'find' such objects. Cf. Hellman [1989b] for examination of some expressive limitations of such a framework. It should be noted, however, that not all constructivist positions are assimilable to this; in particular, Richman [1996] sketches a very different picture: the objects of constructive mathematics are supposed to be no different from those of classical mathematics, nor are the logical connectives different connectives from the classical ones; rather, it is just that the constructivist will not use the Law of the Excluded Middle in proofs. Whether such a view is ultimately coherent and workable are questions beyond the scope of this paper. In any case, the view is *prima facie* not subject to the philosophical critique developed below.

constructive and non-constructive mathematics' (Beeson [1980], p. 22). Moreover, as one considers the more theoretical side of applications, e.g. in quantum theory, greater reliance on what appear to be essentially non-constructive methods emerges.[2] It remains an open question just how far constructivization programmes can be pushed; and so the viability of constructive mathematics as an autonomous substitute for classical mathematics hangs in the balance.

From this perspective, spacetime physics would seem a testing ground of particular relevance. Unlike physical theories set in a mathematical framework of state-spaces, such as Newtonian mechanics, quantum mechanics, or classical and quantum statistical mechanics, in which states themselves are in a sense 'constructed objects' and so *prima facie* may appear open to a constructivist mathematical treatment (see, however, fn. 2), the application of continuum mathematics to spacetime structures is immediate and direct. Newtonian time, for example, is modelled directly as $\mathbb{R}^1$, Newtonian space as (a structure built on) $\mathbb{R}^3$, Minkowski spacetime as (a structure built on) $\mathbb{R}^4$, and General Relativistic spacetimes as manifolds built from charts involving open subsets of $\mathbb{R}^4$. As these theories have been developed, of course, classical mathematics has been employed. Not only have these structures thereby inherited the essentially classical features of the continuum, such as the total linear ordering of $\mathbb{R}^1$, but, as will be emphasized below, the whole modern conception of spacetime is one that may be called 'manifold substantivalism' (following Earman [1989]): spacetime is not thought of as merely a tool we have invented for describing the relative positions and motions of bodies, but rather it is conceived as a genuine, objective physical entity in its own right, supporting matter and metric fields with a physical impact inseparable from that of the galaxies themselves. Thus, the question naturally arises, how is spacetime physics based on a constructivist conception of the continuum supposed to work? What structural features would be altered and what difference would it make? How does a constructivist mathematical treatment comport with the 'manifold substantivalist' conception just alluded to? *Prima facie* a Leibnizian relationist view of space and spacetime would seem more friendly to constructivism, since spacetime itself is viewed as a construction, a device for keeping track of geometric and kinematic relations among actual bodies. Is constructivism in mathematics really tacitly committed to Leibnizian relationism in some form? Is there, moreover, a relationist framework which can shield mathematical constructivism from the charge of 'apriorism', the charge that it imposes limitations *a priori* on objective physical reality which simply need not conform to strictures of mentalist-based pure mathematics? If

[2] See Hellman [1993a, b] for some obstacles to constructivizing mathematics for quantum mechanics. For critical discussion, see Bridges [1995], and for a rejoinder, see Hellman [1997].

not, is there *any* way for constructivism to overcome such an objection? These are our main questions. We do not pretend to provide final, definitive answers here; but our purpose will have been served if a good start can be made, one which may stimulate further inquiry into these and related issues.

2 The constructive continuum: some key properties problematic in connection with spacetime applications

Here we review some basic properties of the constructive continuum which *prima facie* present problems in connection with applications of mathematical analysis to spacetime physics. Discussion of the problematic nature of some of these properties will be postponed until the next section. To focus matters, we will confine ourselves to three types of properties of the constructive real number system and continuous functions thereon, properties pertaining to *order*, *connectivity*, and *extremization*. (The reader already familiar with these features may wish to skip to the next section.)

At the outset, it should be recalled that constructive mathematics begins with familiar construction of the natural numbers, from a starting point iterating a successor relation, this serving as a paradigm for mathematics generally. Furthermore, the rationals are taken as unproblematic, and can be introduced along familiar logicist lines from the natural numbers if a strict logical construction is desired. In particular, the classical order properties of the rationals carry over, e.g. density and trichotomy, the latter being expressed as

$$q_1 < q_2 \lor q_1 = q_2 \lor q_1 > q_2$$

However, with the real numbers, constructive and classical conceptions diverge, not merely on what properties and relations hold but on the domain of the objects as well. The constructivist does share with classical logicism the undertaking of 'constructing' the reals from the rationals; in this respect both differ from a *structuralist* who takes talk of 'the reals' to be a shorthand for talk of any structure meeting certain *defining conditions*, such as the usual order and field axioms together with *separability* (existence of a countable dense subset) and *continuity* (the least-upper-bound principle). (The difference should not be exaggerated, for indeed the structuralist may welcome constructions based on relatively secure items to help establish coherence or realizability of the defining conditions.) The constructivist takes real numbers as, say, Cauchy sequences of rationals (either equivalence classes thereof or certain canonical ones), but in the definition of 'Cauchy sequence', constructive quantifiers are to be understood. Thus, the condition,

$$\forall k \exists n \forall m_{m>n}(|q_m - q_n| < 2^{-k}),$$

on a constructive reading says that a method of finding n depending on any given k is available satisfying the requirement. (Thus, one should really attach subscripts to the logical symbols to indicate that they carry special constructive meanings distinguishing them from the classical ones. We shall refrain from this to ease the notational burden, but the reader should mentally note the intended interpretation, which should be clear from the context.) One then speaks of 'constructive Cauchy sequences'; if Church's Thesis (CT) is accepted, these become identified with recursive Cauchy sequences, of which there are only countably many. Brouwerian and Bishop-inspired constructivists do not incorporate CT into their mathematics, and, moreover, they prove Cantor's diagonal theorem that the reals are uncountable. Nevertheless, the *classicist* reading Bishop, say, who appeals to the notion of a humanly accessible rule for generating converging sequences defining real numbers, will naturally be tempted to conclude that only a countable fragment of the classical continuum is really ever in play in the constructivist setting. Without pursuing this issue further here, it should be noted that it is the constructive reading of the quantifiers that is at the root of the basic differences in properties of the (recognized) real numbers and functions thereon that distinguish constructive from classical analysis.

(i) **Order relations:**

Given real numbers, x and y, generated by Cauchy sequences of rationals, $<x_n>$ and $<y_n>$, respectively, equality is defined by

$$x = y \equiv^{df} \forall k \exists m \forall j_{j,m}(|x_j - y_j| < 2^{-k}),$$

and the ordering on reals is defined by

$$x < y \equiv^{df} \exists k \exists m \forall j_{j>m}(y_j - x_j) > 2^{-k}.$$

$x \leq y$ is then introduced to mean $\neg y < x$. (Of course, these definitions make sense classically, but the constructive meanings should be borne in mind.)

Given these definitions, the first thing to note is that equality between reals is not a decidable relation: $x = y \vee \neg x = y$ is not a theorem of constructive mathematics. This corresponds to the impossibility of 'dividing the line'. Suppose we could determine, of any real x, whether it equals y or not. Then we could, for example, determine whether the real $<r_n>$ defined by

$$r_n = \begin{cases} \frac{1}{2} \text{ if } 2n \text{ is not a counterexample to the Goldbach conjecture} \\ \frac{1}{2} - \frac{1}{2^k} \text{ if } k \text{ is least} \leq n \text{ such that } 2k \text{ is a counterexample} \end{cases}$$

is indeed $= \frac{1}{2}$ or not, which would in turn decide the Goldbach conjecture. Since we have no method of doing this, we have no method of deciding the equality relation. (Of course, once this problem of number theory were to be solved, it could no longer be used in such an argument as a so-called 'weak counterexample', but then another unsolved problem of similar logical form could be. Such arguments of course do not *refute* the proposition being reduced, in this case $x = y \vee \neg x = y$, but they tell us that we cannot include the proposition as part of constructive mathematics.)

Note, incidentally, that the above construction provides a weak counter-example against the classical least-upper-bound principle. A constructive solution to this—i.e. a method of identifying the least upper bound (or greatest lower bound) of any given non-empty bounded set of reals—would solve the halting problem. As applied to the set $\{\frac{1}{2}, < r_n >\}$ as defined above, it would tell us whether the greatest lower bound is $\frac{1}{2}$ or something smaller, which would solve the unsolved problem of the construction.

For analogous reasons, the law of *trichotomy* for reals is not acceptable, viz.

$$x < y \vee x = y \vee x > y,$$

and neither is

$$x \leq 0 \vee x \geq 0.$$

These statements can readily be shown to reduce a form of the halting problem and so cannot be part of constructive mathematics. (For further details, see Beeson [1980].) Given the intuitionistic logical apparatus on which these statements are built, it should be clear to the classicist as well as the constructivist why these statements cannot be constructive theorems, despite the fact that, read classically, they are easily demonstrable from logicist constructions and could even be taken as axiomatic in a structuralist system.

Still, of course, the constructive ordering < does satisfy some important and useful conditions, in particular

$$x \leq y \,\&\, y \leq x \rightarrow x = y$$

and

$$x < z \rightarrow x < y \vee y < z, \text{ any } y,$$

i.e. < is said to be a 'comparative order'. (The first condition alone defines a 'weak ordering'.)

(ii) **Connectivity:**

A good illustration here is the Intermediate Value Theorem, for instance the special case (known as Bolzano's theorem) which states that any continuous function f defined on the interval [0, 1] satisfying $f(0) = -1$ and $f(1) = +1$ has a zero, i.e. $\exists x(f(x) = 0)$. Classically this is proved easily from the least-upper-bound principle together with the weak order property (see e.g. Apostol [1961], p. 169), but constructively it cannot be proved in this precise form. (A constructive solution to this problem would provide one to the problem $t \leq 0 \vee t \geq 0$, which, as we have seen, is unacceptable (cf. Beeson [1980], pp. 11–12). On the other hand, there are constructive *versions* of the Intermediate

Value Theorem, indeed of two sorts. There is, first, a *weakened conclusion* version, that every continuous f as above can be found to take on a value as close to 0 as one pleases, i.e. for any such f

$$\forall\varepsilon\exists x(|f(x)| < \varepsilon).$$

(See e.g. Troelstra and van Dalen [1988], p. 293.) Second, one can prove a *strengthened hypothesis* version. Let $x \# y$ (read *x is apart from y*) be defined as $x < y \vee y < x$, i.e. $< x_n >$ is eventually bounded away from $< y_n >$. Next write $C(f)$ to abbreviate the condition that for any two points x and y in the domain of f, there is a z such that $x < z < y$ and $f(z) \# 0$. Then it can be proved constructively that any continuous function f, as above, such that $C(f)$ has a zero, i.e.

$$C(f) \rightarrow \exists x(f(x) = 0).$$

(See Bridges [1979], p. 30; cf. Troelstra and van Dalen [1988], p. 294.) Since important classes of functions, e.g. *real-analytic* functions, satisfy the condition C, this is a very useful result, in effect an important special case of the classical result. As Beeson points out, this illustrates nicely the way in which classical theorems split into two or more corresponding constructive theorems.

(iii) **Extremization:**

An important classical theorem in analysis, the 'Extreme Value Theorem', states that a continuous function on a compact domain assumes its maximum (minimum) value at some point. As a special case, any continuous real-valued function on [0, 1] assumes its absolute maximum (minimum). This theorem is essentially non-constructive, requiring for its proof the least-upper-bound principle or a related non-constructive property (e.g. Cauchy or sequential compactness, the Bolzano–Weierstrass theorem, etc.). Indeed, one proves that this problem—to find the point at which the extreme value is assumed—reduces the problem, $x \leq 0 \vee x \geq 0$, which, as we have noted above, in turn reduces a version of the halting problem (cf. Beeson [1980], pp. 10–11). Here, as in the case of the Intermediate Value Theorem, there are constructive substitutes, but they are essentially weaker, in this case significantly so. One first replaces the hypothesis of (mere) *continuity* with *uniform continuity*; classically, but not constructively, one proves the famous result that continuous functions on a compact set are uniformly continuous (see, for example, Dieudonné [1969], p. 60, Theorem 3.16.5); constructive mathematics bypasses this by simply *assuming* uniform continuity when it is needed (what the constructivist might concede to be 'theft over *dis*honest toil'). Then it can be proved constructively that if f is uniformly continuous, the supremum (infimum) of its range can be computed, and, hence, that it is approximated arbitrarily closely by values of f, but it cannot be proved constructively that it is actually taken on at any point (see e.g. Troelstra and van Dalen [1988], pp. 294–5). As Beeson points out, in analysis one frequently considers compact function spaces and continuous functionals thereon such that the point at which an extreme value is taken on is the solution of a differential equation or of a variational problem: 'The fact that we cannot find the solution is a major difficulty in constructivizing this branch of mathematics [calculus of variations]—a branch conspicuously absent from Bishop's work and that of his followers, and ripe for constructive treatment' (ibid.). Indeed, in the next section we shall call attention to a generalization of the Extreme Value Theorem which is used crucially in the proofs of spacetime singularity theorems.

3 Spacetime applications: the mathematical challenge (or 'It's only a matter of spacetime')

The mathematical challenge can be easily stated: to find adequate constructive substitutes for classical analytic properties and theorems applied in spacetime physics and physical geometry. Assessing the situation is considerably less easy, however, in part because of the loaded terms 'adequate substitute' and even 'physics', whose interpretation can turn on philosophically weighty matters such as scientific realism versus varieties of empiricism, instrumentalism, etc. Clearly the narrower the conception of 'physics' or 'science' generally, the easier it will be for a constructivist mathematics to 'meet the needs of the science'. But this complexity need not deter us; for the case can be explored conditionally, and one can ask: on the assumption that 'physics' includes 'theoretical physics' and that a largely realist interpretation of physical theory is to be taken seriously—following, say, the modest principles of Hellman [1983] or the closely related 'homely line' Arthur Fine chose not to call 'realism' but dubbed 'NOA' instead ('Natural Ontological Attitude', Fine [1986])—can constructive mathematics recover enough? Should it emerge that constructivism in mathematics is tacitly committed to a strong form of empiricism, instrumentalism, or a similar anti-realism concerning physical theory, that in itself would be an interesting and important conclusion.

Beginning with the most elementary properties of space, time, and spacetime, surely it is taken for granted in classical physics that these are continuous manifolds in the classical sense, and, in particular, that time is totally linearly ordered (i.e. satisfying trichotomy), that spatial lines are so ordered, that particle trajectories are also, and so on. Indeed, it was historically the need to describe such manifolds with mathematical precision that led to the development of classical analysis in the first place. If, on the constructivist conception, the continuum is not even totally linearly ordered, how can the most elementary presumed properties of space, time, and spacetime be described mathematically?[3]

Now the liberal constructivist, who allows for peaceful coexistence, may concede that the classical spacetime conceptions are legitimate, that they cannot be assumed to be fully represented by the constructive continuum, and that, perhaps, at this point, the classical conception must be brought in. Constructive analysis would still apply in part and might provide insight into constructive properties of spacetime continua, even 'empirically constructive properties', to the extent that there is an analogy between mathematical methods of 'finding' or 'constructing' real numbers or values of functions, on the one hand, and empirical methods of approximating instants, spatial points, or spacetime points, on the other. But what about the radical constructivist who rejects non-constructive classical analysis as incoherent or deficient in meaning and, in any case, doesn't allow for peaceful coexistence?

Such a constructivist would presumably put the question to the classicist: 'How do you *know*, for example, that moments of time are totally linearly ordered, or that spatial paths are, etc.? Certainly such things cannot be *proved* [i.e. constructively proved, without LEM]. The very reasoning that blocks us from adopting trichotomy for the real numbers carries over to spacetime manifolds; and similarly for other illegitimate classical principles such as the least-upper-bound principle. The fact that historically such classical properties first arose in the context of space and time, and influenced the development of mathematical analysis, can in no way override the philosophical critique of those ideas based on considerations of meaningfulness and the very possibility of communication' (as set out, for example, in the writings of Michael Dummett, especially Dummett [1977]). The radical constructivist thus 'bites the bullet' and is even willing, should it turn out to be necessary, to say, 'So much the worse for physics!' It may be hoped, of course, not to be necessary: the constructivist may seek to *replace* mathematical physics as we know it with a thoroughgoing, constructive mathematical physics, and claim that the replacement is adequate to any legitimate purpose. It is noteworthy, however, how undeveloped such a project has remained since Brouwer.

In response, the classicist can reply: 'How do we know that the ordering of temporal instants (in the setting of classical physics, for simplicity) is dense, or that it is Archimedean? Even if we confine ourselves to instants labelled by rational numbers, these properties are not known by any sort

[3] 'Presumed properties', we say, because we do not really wish or need to rule out discrete spacetime, for example. Indeed, we are sympathetic to the philosophical stance taken on this question by Forrest [1995]. As will emerge below, what matters ultimately in the present context is not which hypotheses about spacetime happen to be factually correct, but which formulate conceptually coherent possibilities that science needs to be free to entertain.

of direct empirical testing. But from the standpoint of constructive mathematics, they are unproblematic. This already shows that, whatever their superficial similarities, knowability-through-constructive-mathematical-proof and knowability-through-direct-empirical-testing are two quite different things. It may be conceded that none of these properties of spatio-temporal orderings (of instants or points) is known directly empirically. What matters, rather, is the meaningful applicability of classical mathematical structures as useful, perhaps indispensable, models of spatio-temporal reality. Once such models are in place, the empirical success and fruitfulness of the relevant branches of physics can provide a kind of indirect confirmation of presuppositions concerning the mathematical structures exemplified by space, time, or spacetime.[4] The very same can be said of *constructivist* mathematical structures that might be applied to spacetime. Thus, once the *meaningfulness* and *coherence* of classical mathematical presuppositions are granted, the game is over in this stage of the debate between the radical constructivist and the classical mathematician. The question of empirical knowability, however interesting in its own right, is a red herring in this context.'

There is, moreover, a further, equally telling point the classicist can make, namely that the constructivist reasons for resisting, e.g., trichotomy for real numbers are simply irrelevant to the question of trichotomy as a relation among temporal instants or spatial or spatio-temporal points. There are two aspects to this. First, there are the constructive quantifiers and sentential connectives with their distinctive constructive meanings built into the very constructive statement of 'trichotomy'. The classical statement of trichotomy uses classical quantifiers and connectives; when it is asserted, for example, that

$$x \neq y \rightarrow x < y \vee y < x,$$

it isn't claimed that, given the antecedent, there is a method for finding that one of the relations on the right holds, that $< y_n >$ is eventually bounded away from $< x_n >$, above or below. It is merely asserted that one of these relations happens to hold *in fact*, even if we have no method of 'telling' or 'proving' which. This distinction—between distinct *laws* of trichotomy, intuitionistic and classical—pertains both to real numbers, construed as (equivalence classes of) Cauchy sequences of rationals, and to the spatio-temporal point objects that may be represented by reals. Second, and related, when the constructivist presents an argument by counterexample to trichotomy, understood

[4] As Weyl put it so well, in the continuation of the paragraph from which we quoted above: 'The propositions of theoretical physics, however, certainly lack that feature which Brouwer demands of the propositions of mathematics, namely that each should carry within itself its own intuitively comprehensible meaning. Rather, what is tested by confronting theoretical physics with experience is the system as a whole' (Weyl [1949], p. 61). This view, of course, became a central theme of Quine's and is at the core of one kind of 'indispensability argument' associated with Quine and Putnam.

intuitionistically, or to decidability of = between reals, etc., what it shows, from the classicist perspective, is that we have no constructive method of telling *which* spatio-temporal point is *represented by* a certain rational sequence *as presented*, i.e. in the counterexample. Even if we agree to define real numbers by convergent rational sequences, the classicist can resist *identifying* such a sequence with its *presentation* or *specification*; and, moreover, she will certainly resist identifying such a *presentation* or *rule for generation* with a spatio-temporal object. Of course it may not be known of certain *specifications* of real numbers which spatio-temporal points they *represent*, under some given scheme setting up a representing homomorphism between reals and points. Whatever implications this may have for ordering relations among *constructive* reals considered as given by such specifications, this has nothing to do with an *objective order relation* among the spatio-temporal points themselves.

This raises some philosophical questions to be pursued further in the next section. But what of theorems of spacetime physics? Is the restriction to constructive logic, i.e. intuitionistic logic, really any hindrance? A full answer to this question is beyond the scope of this paper, but let us call attention to one topic whose standard treatment is essentially non-constructive, but which arrives at some of the most striking results in general relativity and cosmology, namely the spacetime singularity theorems of Hawking, and Hawking and Penrose. For full details, the reader should consult Wald [1984] and O'Neill [1983]. Here we will just mention the main points relevant to the problem of constructivization.

The spacetime singularity theorems fall within the topic of causal structure of Lorentz manifolds. A central question which plays a crucial role in the proofs is the existence of length-maximizing geodesics in manifolds satisfying various specified conditions, e.g. absence of closed, causal curves, compactness of sets containing all (future-pointing) causal curves from p to q, etc. It is exactly here that problems of essential non-constructivity arise, for what is involved is a generalized form of the Extreme Value Theorem, reviewed above.[5] Wald's treatment brings this out clearly, as with the theorem numbered 9.4.4 (pp. 236–7):

Theorem (9.4.4) *Let (M, g_{ab}) be a globally hyperbolic spacetime (equivalently, possessing a Cauchy hypersurface). Let $p, q \in M$ with $q \in J^+(p)$ (i.e. q lies in the causal future of p). Then there exists a curve $\gamma \in C(p, q)$ for which the length function τ attains its maximum value on $C(p, q)$. [$C(p, q)$ is the set of continuous, future-directed causal curves from p to q.]*

[5] For conciseness, we are bypassing applications of the Intermediate Value Theorem. As indicated, however, reasonably good constructive versions of this are available, whereas the applications of the Extreme Value Theorem to be described remain a substantial challenge for constructive mathematics.

The proof proceeds in two non-constructive steps. In the first, previous developments are cited for the compactness of $C(p, q)$, going back ultimately to the Bolzano–Weierstrass theorem. (Here one needs a subtheory of limits of sequences of causal curves, developed in O'Neill [1983] as 'quasi-limits' (pp. 404ff.). This whole subtheory is non-constructive, depending repeatedly on the Bolzano–Weierstrass theorem or, equivalently, sequential compactness.) Then the upper semi-continuity of the function τ is cited and, in the second non-constructive step, the Extreme Value Theorem for such functions on a compact domain is invoked. Wald then presents an exactly analogous theorem, 9.4.5, in which the point p of 9.4.4 is replaced by a Cauchy surface Σ.[6]

These theorems, together with previous theorems placing limits on how far length-maximizing geodesics between a hypersurface and a point can be extended, are then used to prove the singularity theorems (9.5.1 and 9.5.2 in Wald, 55A and 55B in O'Neill).[7] As Wald puts it:

> [These and two further theorems] establish the existence of singularities in the sense of timelike or null geodesic incompleteness under conditions relevant to cosmology and gravitational collapse .... [The first, due to Hawking] can be interpreted as showing that if the universe is globally hyperbolic and at one instant of time is expanding everywhere at a rate bounded away from zero, then the universe must have begun in a singular state a finite time ago (Wald [1984], p. 237).

The second theorem, also due to Hawking (9.5.2 in Wald, 55B in O'Neill), removes the assumption of global hyperbolicity, replacing it with the assumption of a compact, spacelike hypersurface, and proves less, viz. that there exists at least one inextendable, past-directed, timelike geodesic from the compact hypersurface with length no greater than a specified bound. Furthermore, a still more widely applicable theorem, due to Hawking and Penrose [1970], is stated (as Theorem 9.5.4 in Wald [1984]), in which both the assumptions of global hyperbolicity and of expansion everywhere are eliminated, although no information is provided concerning which timelike or null geodesic is incomplete. The proofs of all these theorems rely heavily on the existence of length-maximizing geodesics (9.4.5, above) and sequential compactness. They establish geodesic incompleteness of any Einsteinian spacetime meeting the surprisingly general stated conditions, which amount to little more than gravitational attraction, the absence of closed timelike curves, and past (or future) convergence of a suitable geodesic congruence. The theorems do not tell us

6 The essential non-constructivity of this theorem, which can be illustrated by weak counterexample, should be compared with the simpler but analogous result on the surface of a sphere, where the problem of proving the existence of a length-minimizing geodesic between arbitrary given points is essentially non-constructive (cf. Beeson [1980], p. 20, Exercise 1).

7 For a clear overview of the *reductio* structure of the argument leading to spacetime singularities, see Wald [1984], p. 212. A thoughtful discussion of the singularity concept follows, and then the mathematical details are clearly set out in the remainder of the chapter (9).

how to construct an inextendable, finite-length geodesic; and the first does not even arrive at a contradiction constructively from the assumption that not every past-directed timelike curve from the Cauchy hypersurface has length within a certain bound. Nevertheless, these are physically highly significant theorems establishing the existence of striking anomalies in a surprisingly broad class of spacetimes, not just those of a highly symmetric character, such as the Robertson–Walker spaces.[8] Our own universe may well be covered by the more general theorems. Indeed, as Wald puts it, 'Theorem 9.5.4 gives us strong reason to believe that our universe is singular .... Thus, it appears that we must confront the breakdown of classical general relativity expected to occur near singularities if we are to understand the origin of our universe' (Wald [1984], p. 241).

All this raises two questions. The first is simply whether constructive mathematics can find reasonable alternative formulations of such theorems and prove them constructively. The conclusions of such theorems should presumably provide more information than the classical ones—it would undeniably be good to know 'how to find singularities'—but offsetting stronger hypotheses are to be expected. The second question is whether the constructivist should bother to try. Here differing opinions are certainly possible, depending in part on one's conception of 'physics' and the role of mathematical models of physical reality. On the one hand, a scientific realist is inclined to take an empirically successful theory seriously even in its highly theoretical aspects. In the present case, surely it is important to learn that classical General Relativity breaks down in certain cosmological and gravitational-collapse contexts, that certain *prima facie* appealing cosmological models, such as a non-singular 'bouncing universe', are ruled out by certain reasonable energy conditions and the other spacetime conditions needed for the singularity theorems. On the other hand, it is possible to stand back from the singularity theorems and view them as merely a kind of commentary on our theories and models, and not really 'part of physics'. (Indeed, one of the leading contributors to the subject of models of GTR and singularity theorems, Robert Geroch, has expressed views along these lines.)[9] If one holds the mathematical constructivist to the task of recovering only that mathematics which leads directly to testable empirical results, constructivizing such results as the singularity theorems may appear *recherché* or as a purely recreational enterprise. Without attempting to resolve this dispute here, let me suggest that, regardless of how one wishes to use the honorific term 'physics', there is no easy way to separate learning about models of successful theories—which we may have good

[8] It should be pointed out that, in the case of Robertson–Walker spaces, a cosmological singularity theorem can be proved constructively. (It is possible to give a constructive proof of, e.g., Proposition 15 of O'Neill [1983], p. 348.) However, the theorems of Hawking and Hawking and Penrose highlighted here are of far greater generality, applicable to a much wider range of spacetimes, and hence of greater significance.

[9] In personal communication.

reason to believe are at least partly, probably, approximately true—from learning about physical reality. From this perspective, there is a strong case for holding the mathematical constructivist to the higher standard: mathematics must be rich and flexible enough to allow entertaining the widest range of conceptual possibilities, for we simply cannot say in advance just which kind of universe we occupy. We will continue with variations on this theme in the next section.

4 The broader philosophical challenge

There is a general philosophical problem confronting radical constructivism which began to emerge when we considered spatio-temporal ordering relations above. It arises as follows. Recall that radical constructivism conceives of mathematics as a theory of computational procedures and objects. We can say it is *agent-oriented*: it is concerned ultimately with the mental capacities of an idealized human mathematician. As Bishop put it: 'When a [hu]man proves a positive integer to exist, he should show how to find it. If God has mathematics of his own that needs to be done, let him do it himself' (Bishop [1967], p. 2). Thus, the 'principle of omniscience'—that all elements of a set A have a property P or some element of A lacks P—has no place in constructive mathematics (and so, for the radical constructivist, no place in mathematics). Thus, 'constructive existence is much more restrictive than the ideal existence of classical mathematics. The only way to show that an object exists is to give a finite routine for finding it ...' (Bishop [1967], p. 8). As Dummett concedes, however, such a view is questionable in connection with objects in general, say stars. Closer to home, forensic evidence may convince us beyond any reasonable doubt that a person died due to a homicide, hence that a killer exists, without any method of finding the killer being provided or providable. (Maybe the mystery is genuinely unsolvable.) But, Dummett says, such considerations are irrelevant to mathematics; for example, the classicist's point that 'there is no absurdity in thinking of an infinite totality [say of physical objects] as already formed ... cannot be applied to mathematical totalities, whose elements are mental constructions' (Dummett [1977], p. 58).

Now the problem should be clear: in physics and other sciences, we frequently employ mathematical structures as models of some parts or aspects of the physical world; but if those structures are constrained by constructivist restrictions, based on the mentalist conception just described, what reason is there to suppose that objective physical structures will be captured? Why *must* functions representing physical magnitudes necessarily be constructive functions? Why *must* relations, such as order relations, be constrained only by constructively correct conditions (e.g. comparative order, as opposed to total order)? Why *must* instants or points be representable by constructive real numbers, as opposed to arbitrary real numbers? To insist that such restricted representations are required by human thought would be to commit apriorism no less egregiously than did Kant, as we can now say with hindsight,[10] when he argued that space and time 'must' be Euclidean lest we not be able to form a coherent conception of a located material object. The point can also be put in a somewhat different way. Grant, for the sake of argument, the constructivist view of *pure* mathematical postulates, that they should have the 'evident' character and certitude of constructive truth. Why should these standards carry over to *applied* mathematical postulates, i.e. to scientific *hypotheses* concerning physical structures mathematically described? After all, no one expects certitude of such hypotheses; we consider ourselves lucky if we can just achieve empirical adequacy. Why should there be any restrictions *a priori* on the character of the mathematics that may be used to describe real or idealized physical systems?

One answer can be found in writings of Dummett: to transcend the bounds of intuitionistic logic is to transcend the bounds of sense; to succeed in communicating with language, something analogous to the intuitionistic proof-conditions must be understood, not classical truth conditions, which are seen as inaccessible (see e.g. Dummett [1991]). This line of argument, of course, applies across the board to any subject matter, not just mathematics. To this writer (and as Dummett seems to concede), this is in effect a revival of the verification theory of meaning; but it seems subject to much of the many-faceted critique of that theory that has been developed over several decades in the middle of this century. Recall briefly some of the salient points brought out by that critique: a hidden reliance on counterfactuals (a point conceded in Dummett [1991]); the relativity of testing to background assumptions (confirmation being at least a three-place relation, not simply a two-place one between hypothesis and evidence); and, generally, the inability of verificationism to explain much linguistic functioning as it is ordinarily described.[11] Confronted by this, the radical constructivist may adopt the more modest tack alluded to above, conceding that *physical* objects and structures *need not* in fact conform to constructivist strictures, but still insisting that *mathematical* objects and structures must.

[10] Whether Kant's view was a defensible one given the science of his day is a question we can safely bypass here.

[11] It is just here that circularity threatens on all sides. If objects of a certain type (e.g. unobservable physical objects such as atoms and their constituents, not to mention mathematical abstracta) are taken for granted, and it is therewith assumed that indeed we *do* succeed in referring to such things with language, then this is a kind of functioning that an adequate meaning-theory must account for, and for well-known reasons verificationist accounts face apparently insuperable obstacles. If, on the other hand, an anti-realist position with respect to such putative objects is adopted at the outset, then there is no such reference for a theory of meaning to account for. Despite this evident situation, a persistent theme of Dummett [1991] is that a theory of meaning based on 'use' must somehow be developed prior to any 'metaphysical' assumptions. On our view, which cannot be argued for here, any such attempt to reinstate a 'first philosophical' theory of meaning prior to all science is doomed.

This, however, will simply not do. Mathematics of the continuum, after all, can be developed in such a way that mental objects—and even abstract objects, if that is the constructivist's worry—are entirely avoided: in fact, all one need do is entertain the *logical* possibility of sufficiently rich *physical* structures. Indeed, the possibility of a discrete infinity of atomic individuals, together with arbitrary wholes formed from them, suffices to recover the vast bulk of scientifically applicable mathematics (cf. Hellman [1989a, 1996]). Now, the radical constructivist who grants that reasoning about even infinite physical structures may obey full classical logic faces a little dilemma: on the one hand, if it is also conceded that at least this much pure mathematics (i.e. scientifically applicable mathematics) *can* be conceived and developed classically, then radical constructivism collapses to a liberal variety, admitting peaceful coexistence. This avoids the apriorism charge ('wins the battle') but at the cost of renouncing the radical position ('losing the war'). Alternatively, if somehow this concession is resisted, to allow that physical objects and structures need not conform to constructivist strictures, while insisting nevertheless that *mathematical* objects and structures must, is to leave the radical constructivist in a self-defeating position: for in effect it has been conceded that there *is* no *a priori* reason why constructivist mathematics ought to suffice for physical applications, while at the same time a richer alternative has been disallowed. No wonder classicists resist reformation.

What this shows so far, I believe, is that radical constructivism which grants the coherence of classical reasoning about the physical is inherently unstable. To preserve a viable identity, it must instead acknowledge continuity between mathematics and physics on the question of what type of logical reasoning is coherent, that is, it must insist upon *intuitionistic logic only* for both. But then it is driven back to confronting the apriorism charge all the more starkly. For now, in addition to the typical questions concerning mathematical modelling raised above in describing that charge, there can be added more general questions such as, 'Why should it not even be logically possible that infinitely many particles actually exist with determinate properties not "decidable" by us, whatever "decision methods" are allowed?' Whether somehow this charge can be averted or circumvented will be discussed further in the final section below.

To conclude this section, we should consider a recent argument that constructive mathematics à la Bishop is not really more restrictive in applications than classical mathematics. Were it correct, it would undercut much of the force of the arguments of this paper. It runs thus: 'You have been arguing as if constructive mathematics is *more restrictive* than classical mathematics, and indeed this is a view commonly taken by classicists trying to understand constructive mathematics, and it is suggested by some constructivists' remarks. However, there is a clear sense in which Bishop's constructivism is *more general* than classical mathematics: since it is characterized by classical logic less the LEM (or anything implying it), every constructive theorem is also a classical theorem. The difference is that, since LEM is not used, constructive theorems hold in a wider class of structures, including not only any classical structures but also structures in which the logical apparatus receives a constructive interpretation. So while it is true that constructive proof demands "more" than classical proof, the applicability of constructive results is more extensive than that of classical results' (this paraphrases Richman [1996]; see also fn. 1.)

This is an interesting logical point, and it indicates one clear advantage of Bishop's approach to constructive mathematics over intuitionism: by working with a proper subset of classical axioms and not adding new ones, such as the continuity principles in the theory of choice sequences, all results obtained in the Bishop framework are classically correct, but open to a constructivist interpretation. However, this increased generality—call it '*logical* generality'—is offset in two significant ways. Firstly, it is offset by the need for stronger hypotheses and/or weaker conclusions in theorems of conditional form (which are typical): since the logic is weaker, these adjustments are needed to take up the slack, unless the classical theorem is constructively valid as it stands. Schematically, the situation can be depicted as follows, where L_{Co} stands for '*core logic*' = *constructive logic*, $A^{(+)}$ is the hypothesis of a theorem, and $C^{(-)}$ is the conclusion:

Classical	Constructive
L_{Co}	L_{Co}
$+LEM$	
A	A^+
$\downarrow$	$\downarrow$
C	C^-

While the constructive conditional theorem, $A^+ \rightarrow C^{(-)}$, can indeed generally be given more interpretations of its logical vocabulary, in another sense—of far greater interest in scientific applications of mathematics—the classical theorem is 'more general' in that its *conclusion*, C, is proved to hold in a wider variety of situations to the extent that A^+ is a stronger hypothesis than A. (One may call this 'generality of *conclusions*'.) Thus, in the basic example of the Intermediate Value Theorem, the classicist proves that intermediate values are taken on by any continuous function on a compact set, whereas the constructivist proves this only for restricted subclasses of functions, e.g. the real-analytic ones. This is typical of a great many constructive theorems. And, if our assessment above is sound, the constructivist will be able to prove geodesic incompleteness at best for a restricted class of spacetimes, as compared with the Hawking or Hawking and Penrose theorems. In general, in scientific applications of mathematics, the goal of explaining and understanding natural phenomena is paramount, not achieving a constructive interpretation of results, and so from this perspective, we submit, it is this second kind of generality that really matters.

Secondly, moreover, it is not just that stronger hypotheses are often needed by the constructivist. Whole axiom-systems are at stake. Consider the axioms for a complete, separable, ordered continuum, for example; classically, this includes the least upper bound principle or something strong enough to imply it, and, of course, it will specify a total, linear ordering. Now such an axiom system may be understood in two quite distinct ways, either as *descriptive* of some already recognized domain, or as *implicitly defining* a type of structure of interest. Taken in the first way, to be constructively acceptable the axioms have to be *constructively true* over an intended domain, which in this case they surely are not. Taken in the second way, presumably the constructivist will require a constructive proof that such structures exist or are possible. In this example, no such proof is available, so even as an implicit definition—though quite possibly intended to introduce models for physical applications—such an axiom system is not constructively acceptable. Thus, the constructivist approach to axiomatics is indeed much more restrictive than the classical, and so the charge of an unjustifiably restrictive apriorism *vis-à-vis* scientific applications of mathematics still has its force, in spite of the increased *logical* generality of Bishop-style constructivism.

5 Can Leibnizian relationism help?

As already suggested, the argument of the last section, as it pertains to spacetime, applies in the first instance to the view of spacetime known as manifold substantivalism, the view that accords genuine physical reality to the spacetime manifold itself. Spacetime as a domain of points is what supports matter fields and even the metric itself; it is physically objective as much as are the more commonplace objects and events that occur in spacetime. Of course, in General Relativity, spacetime is not a fixed background like Newton's space; it is a dynamical object which evolves interdependently with energy-momentum. But, if anything, this dynamism helps reinforce a straightforward substantival interpretation of the mathematical structure, a Lorentz manifold, for it makes it difficult to regard spacetime as a 'nomological dangler', an extraneous fiction useful merely for descriptive purposes.

It is, of course, the objective physical status of spacetime that prompts the charge of apriorism levelled at the radical mathematical constructivist who would impose mentalist strictures on spacetime relations and structures. Indeed, the argument of the previous section could be called 'the argument from objectivity of spacetime'. This naturally leads one to ask whether the constructivist can evade this argument by simply rejecting manifold substantivalism outright and opting for a Leibnizian, relationist view of spacetime. One expects that such a view is more congenial to constructivism, since it denies physical reality to spacetime and itself seeks to 'construct' or 'reconstruct' talk of spacetime as merely a convenient device for encoding relations among more palpable things. Provided that *such* relations can be adequately mathematized within constructivist systems—a substantial *if*, to be sure—constructivism would seem to be off the hook and free to counter-charge the platonist mathematician with multiple counts of ... well, *platonism*. As if Cantor's heaven weren't enough, the heavens themselves also are treated as an object, 'above and beyond' (= underlying!) all matter-energy.

On a closer examination, however, matters turn out to be anything but straightforward. The whole question of Leibnizian relationism has recently been examined in detail, taking account of relativistic physics, by Earman [1989], and his treatment serves us as a useful guide.

First, it is necessary to distinguish two relationist theses, frequently conflated: the first, which Earman labels R1, asserts that all motion is relative motion, that it makes no sense, for example, to say of the two isolated spheres of Newton's famous thought experiment, that one of them is absolutely rotating whereas the other is not. This thesis is simply not tenable in the context of our best spacetime physics. In General Relativity, absolute rotation is well defined, and Mach's principle, according to which observable effects of rotation, centrifugal forces, etc. are to be explained with reference to the fixed stars, is simply not respected. Since R1 was used by relationists to motivate the rejection of absolute space, its failure deprives relationism of one of its underpinnings. Of course, absolute *space* is avoided even in classical Newtonian gravitation theory (where Galilean relativity reigns), but 'absolute' *spacetime* is an entirely different story, and, as already indicated, survives in the form of manifold substantivalism as the natural, direct reading of GTR. The second relationist thesis, R2, is a direct denial of substantivalism: spatio-temporal relations among events and bodies are direct, and there is no underlying reality of spatio-temporal points or regions. The appearance to the contrary stems from too literal a reading of the mathematics of differentiable manifolds; such purely mathematical structures are merely a useful device for codifying relations among events, and should not be understood as representing an entity, 'spacetime'. 'Antisubstantivalism' is a more accurate term for this than 'relationism'. And it is this thesis that concerns us here, as a potential ally of mathematical constructivism.

Although R1 fails, Earman finds independent motivation to pursue R2 in an application of the method of Einstein's 'hole argument': because substantivalism regards manifolds related by a diffeomorphism as corresponding to distinct physical possibilities, it turns out to be guilty of its own form of apriorism, namely that it rules against the very possibility of *determinism* in advance (Earman and Norton [1987]). Although substantivalists have responses (which Earman [1989] reviews), this motivates further development of the relationist (R2) program. A persistent theme of Earman's is that a thoroughgoing relationist treatment of GTR simply does not exist, and that it is unclear whether it is even possible. Two approaches are examined, and it is appropriate for us to ask whether either of them can lend any comfort to the mathematical constructivist.

Let us consider first the approach Earman considers second (Earman [1989], Ch. 9, section 10), based on a 'plenum of physical events'. The idea, which derives from Reichenbach's writings, is to take causal relations on a domain $\mathcal{E}$ of pointlike events as primitive and to attempt to recover enough structure to carry out spacetime physics. Earman considers the most straightforward interpretation of this, namely to recover substantival models of the form M, g of GTR, where M is a differentiable manifold and g a metric tensor of Lorentz signature. Thus, points are constructed as equivalence classes of events of $\mathcal{E}$, equivalence being defined as bearing the causal relations of timelike, lightlike, and causal precedence to all the same events. A topology is induced on the set of points by taking as a basis sets of the form $I^+(p) \cap I^-(q)$ where $I^+(p)$ is the set of points timelike connected from p, and $I^-(q)$ is the set of points timelike connected to q, where the relation of timelike connectedness is transferred in the obvious way from the relation on $\mathcal{E}$. (Orientability of the manifold is assumed.) Earman points out that this builds in strong causality, violated in some models of GTR, which then would have to be argued to be unphysical. More serious is the circumstance that differential structure has yet to be recovered. But even if this can be done, by introducing enough machinery on the set $\mathcal{E}$, there is still the 'whiff of circularity' involved in positing a plenum of events which, presumably, are in many cases nothing but the assuming of (exact) field values, and, since the g-field, at least, is everywhere defined, this means that to every manifold point there corresponds such an 'event'. In the present setting, in any case, 'whiff' is clearly a euphemism: the plenum $\mathcal{E}$ is posited as a mind-independent physical entity as much as is the substantivalist spacetime manifold; its pointlike objects and the relations they bear to one another are just as much prior to and independent of computing agents' capacities as are spacetime points and their relations. Whatever force the aprioricity charge has in its spacetime substantivalist formulation carries over *mutatis mutandis* to the corresponding charge regarding the posited cosmic plenum. We conclude that this way of sustaining Leibnizian relationism (R2) is of no comfort whatever to the radical mathematical constructivist.

A more promising approach is the first one considered by Earman. This begins with explicit statements in authoritative presentations of GTR and spacetime structure (e.g. Hawking and Ellis [1973]; Wald [1984]) that a model (M, g) represents physical reality only up to a diffeomorphic transformation d of M—that is, that a representation (M, g) and another (M', g') are physically equivalent if some d is an *isometry* between them, i.e. d is a bijective map between M and M' preserving manifold structure and $g' = d * g$, the 'drag-along' of the metric tensor g. Thus, we should really think of equivalence classes of mutually isometric models as 'representing space-time reality'; we work with an individual representative of the class for convenience, but any isometric model is just as correct. Thus, a direct substantivalist reading of a presentation of GTR, which construes the elements of a single M as spacetime points, is resisted from the start. There is an analogy with structuralism in mathematics: one refers, for example, to a unique domain of natural numbers with a unique successor relation, etc., defined on it, but this is only a convenient mode of presentation: really one intends to be describing a mathematical reality only up to isomorphism.

Realizing relationism, however, involves a further, positive step, Earman suggests (Earman [1989], Ch. 9, section 9): one should provide a manifold-independent means of expressing objective spacetime structure. Here one can have recourse to a method of Geroch [1972] which can be read as realizing a suggestion of Einstein's, that spacetime, M, 'does not claim an existence of its own, but only as a structural quality of the [gravitational] field' (Einstein [1961]). Given a C^{∞} manifold, M, one defines various rings of functions on M such as $C_0(M)$, the ring of continuous real-valued functions on M, $C^*_0(M)$, the subring of bounded continuous functions, and $C^{\infty}(M)$, the ring of C^{∞} real-valued functions on M, and $C_c(M)$, the ring of constant functions (isomorphic to **R**). Next one shows how to code the various vector and tensor fields needed to carry out GTR in terms of mappings between these rings and objects constructed from them. A contravariant vector field, for example, can be defined as a mapping V from $C_{\infty}(M)$ to $C_c(M)$ satisfying $V(\lambda f + \mu g) = \lambda V(f) + \mu V(g)$ and $V(fg) = fV(g) + V(f)g$, for $f, g \in C_{\infty}(M)$ and $\lambda, \mu \in C_c(M)$. Covariant tensor fields can then be characterized as multilinear maps from tuples of contravariant vector fields to $C_c(M)$, and so on. Finally, one 'throws away' the manifold M and keeps just the algebraic structure, called an 'Einstein algebra' by Geroch or a 'Leibniz algebra' by Earman. Different realizations of such an algebra are physically equivalent and can be thought of as giving different representations of the same physical reality. Spacetime as an underlying object in its own right has been eliminated, at least in so far as one views the initial M as merely a heuristic device leading up to the Einstein or Leibniz algebra.

Now, as it has just been sketched, this approach cannot rescue constructivism from the aprioricity charge for the simple reason that the Leibniz algebra is based on substantivalist models classically described. That we are dealing with a plethora of mutually isometric such models—that is, sifting out what is common to all of them—rather than a single such cannot help constructivism, since all of them will share many essentially non-constructive properties; indeed they will share any property that is determined by differential *cum* metrical structure. Consider, for example, the essentially non-constructive properties crucial to the Hawking singularity theorems noted above, viz. the compactness of $C(p, q)$, the space of continuous causal curves from p to q, in globally hyperbolic space-times, which in turn guarantees the existence of length-maximizing geodesics in $C(p, q)$.[12] These are stated with respect to a particular manifold, of course, but they are preserved under isometric transformations and so would presumably show up in the more abstract setting of a Leibniz or Einstein algebra. If anything, this only reinforces the apriorism charge, since it now emerges that taking certain *properties* of 'spacetime' as objective, physical properties does not ultimately depend on taking spacetime itself as a genuine object. In Einstein's language, what matters is 'a structural quality of the [metric] field', which can be represented mathematically in many equivalent ways. It does not thereby lose its physicality and become 'merely a mental construction'.

Where does all this leave the radical mathematical constructivist? Some options remain, before throwing in the towel. One—a relatively conservative one—is to try to carry out the Leibnizian strategy just described but with *constructive* models of spacetime structure rather than (mathematically) classical ones. The resulting *constructive Leibniz or Einstein algebra* would then be taken as capturing objective physical spacetime properties and relations, and these would simply not include all the classical ones, such as the existence of length-maximizing causal geodesics under certain conditions (such as existence of a Cauchy surface). Assuming this could be done, it may appear that there is a standoff: the (mathematical) classicist takes a class, C, of isometrically invariant properties as constituting spacetime structure (without literally treating spacetime as an object), and the constructivist takes a different class, C', of isometrically invariant properties—invariance now being understood over a class of constructive spacetimes—as constituting spacetime structure (also without literally treating spacetime as an object), and it might seem that it is anyone's guess who is right. The question would then seem to be an empirical one in a broad sense. But this is really to misconceive the situation: as already emphasized above, the question is not really whether or

[12] Actually, a good deal more work needs to be done to show that properties arising from parts of a manifold involving variables over spacetime points, such as properties of $C(p, q)$, can be given a satisfactory, purely algebraic characterization. One must first introduce the notion of a 'curve in an Einstein algebra'—something that seems doable by composing functions parametrizing curves with functions in $C_\infty(M)$—and then recover the requisite causal relations, and it is not completely obvious that a fully algebraic reconstruction is possible. If it is not, then the whole discussion is shortened at this point, as this route toward realizing Leibnizian relationism (R2) cannot really succeed anyway. Let us proceed, however, for the sake of argument, on the assumption that the approach really can be adequately developed to cover causal structural properties of the sort we are considering.

not the classicist is empirically or scientifically *correct* in the classical characterization of actual spacetime structure. It may be admitted that perhaps all these descriptions are idealizations and not literally true, perhaps not even approximately true. What matters is that it be a *coherent possibility*, epistemically, that a classical description be correct, and that the physicist be free to put forward classical models as potentially 'best explanations', among other things. Yet it is precisely this that the radical constructivist denies in attacking the intelligibility of the mathematical ideas underlying the classical spacetime models. The classicist, on the other hand, does not deny the coherence of constructivist models, but rather regards them as picking out at best only a proper subset of spacetime structural properties. ('At best' because, on a reading of applied constructive mathematical statements parallel to that of pure statements reviewed at the outset of this paper, all 'structural properties' would really bring in reference to what an idealized agent 'can construct', and so would not even qualify as objective structural properties in the ordinary sense.) In this crucial respect, there is not a standoff but rather something more like a breakdown in negotiations. It is not a situation in which each side says to the other, 'I think my description is better, but it remains possible that yours is', but rather one in which the classicist says something like this to the constructivist whereas the constructivist tells the classicist that she is not even formulating coherent possibilities. Again we are back to the apriorism objection: why should *objective, physical structures* be limited to what can be described within constructivist language or inferred only by rules justified by appeal to that language? One is of course free to practise *pure* mathematics within such confines if one wishes, but to restrict *applied* mathematics in that way is still to invite an adjusted version of Hamlet's famous remark to Horatio: *there may well be more things in spacetime than are dreamt of in your applied mathematics.*

The only real way, it seems to me, for the radical constructivist to get around the apriorism charge, at least as it pertains to spacetime structure, is to take a truly radical line and deny the major premise on which that charge is based, namely that there *is* such a thing as an objective, mind-independent spacetime structure that our mathematical physics seeks to describe. So long as this premise is left intact, some version of the apriorism objection will surface, as we have just seen in the case of the 'conservative' approach to implementing Leibnizian relationism (R2). It is insufficient simply to deny the reality of spacetime points or regions; it is necessary to go further and deny the reality of objective, physical spacetime structure altogether. But if that is done—if it is maintained that all mathematical models in this domain are merely convenient devices for helping us organize our experience and should not be taken seriously even as non-unique, approximating representations of physical reality—then at least the way is open to a thoroughgoing constructivist treatment of this much applied mathematics without running afoul of the apriorism charge. If 'physical spacetime structure' altogether, not just any particular theory of it, is really a myth, or if it is relegated to the status of absolutely unapproachable *noumena*, then the objection dissolves.

We need not go further here by enumerating the deep difficulties we find with anti-realist views of science, nor need we review, from a scientific standpoint, their error-strewn record, from Berkeley through Mach, Reichenbach, and beyond.[13] It is sufficient for us to draw the conditional conclusion, leaving the choice of *modus ponens* versus *modus tollens* to the reader: the radical mathematical constructivist, it seems, must indeed also be committed to a strong form of anti-realism concerning spacetime physics, and, presumably other branches of science as well. The particular form of anti-realism is not hereby precisely specified, but its thrust would be comparable to that of radical empiricism, or of instrumentalism.[14] From an experimental or observational (if

[13] I am referring, for example, to Berkeley's sophistical arguments turning on equivocation against the 'mind-independence' of matter, to Mach's embarrassing rejection of the atomic hypothesis and his account of rotation, and to Reichenbach's continuation of Mach's views based on misunderstandings of GTR. The latter are enumerated and treated in detail in Earman [1989]. For an example of the sort of trouble a more recent, influential anti-realist approach (that of van Fraassen [1980]) runs up against in its effort to draw an absolute yet epistemically weighty dichotomy between the 'observable' and anything beyond (in which rational belief would be unjustified), see Chihara and Chihara [1993]. It should be stressed, however, that van Fraassen's 'constructive empiricism' does not challenge the *meaningfulness* of highly theoretical statements—in this it is at least 'non-destructive'—and so is not sufficiently radical for the Dummettian variety of mathematical constructivism.

[14] Some recent efforts to articulate a 'Dummettian anti-realism' concerning scientific theories seek to improve upon radical empiricism and instrumentalism (e.g. Luntley [1982] and Wright [1993]). The essential idea is to acknowledge the theory-laden character of 'observation', indeed to renounce any epistemologically privileged 'observation language' for science, but to insist, nevertheless, that the concept of *truth* for scientific statements must be 'epistemically constrained', leading to indeterminate truth status for many such statements due to the absence of any available 'decision procedures' for them. Thus, for example, under suitable experimental conditions involving elaborate apparatus, the existence of subatomic particles can be 'empirically decided', at least by suitably trained physicists, in agreement with a realist point of view; nevertheless, to say that the desk on which I am now typing is now composed of such particles is to venture into the 'indeterminate', since the relevant apparatus for performing the requisite physical experiments is not in place. Similarly, even ordinary statements about the past, such as 'Ten years ago to this day there were 126 paper clips on my desk', are 'indeterminate' for similar reasons (such examples are given in Luntley [1982]). While this is clearly not the place to undertake a thorough examination of such a view, I would make two remarks: first, the view radically severs the presumed links between laboratory and non-laboratory circumstances crucial to the explanatory force of much of modern science. For example, it makes a mockery of the whole programme of physics and chemistry to account for macroscopic properties of matter in terms of microstructure and basic forces. For in the vast majority of circumstances in which these theories are supposed to apply, the relevant laboratory apparatus for verifying microstructural claims is not in place, and is not 'available' except in a *counterfactual* sense itself regarded as indeterminate by this version of anti-realism. Second, the view appears to fly in the face of the apriorism objection emphasized in this paper, or at any rate a closely analogous objection: Why should we expect the determinacy of properties of the natural world to turn on what is 'empirically decidable' by us, however this imprecise phrase is spelled out? In response, following out the 'truly radical line' to which such a view seems driven, it may be acknowledged that a result of imposing this standard of determinacy is indeed to renounce the natural world as it is ordinarily and scientifically conceived.

not practical) point of view, applied constructive mathematics may be made to work tolerably well. From a theoretical standpoint, there seem to be serious limitations arising from the non-constructivity of theorems of deep physical significance, such as the singularity theorems. But even if alternative versions of such theorems turn out to be constructivizable, there remains the theoretical difficulty emphasized in this paper, that from a realist perspective regarding physical geometry, there is every reason for leaving open the full range of logical and mathematical possibilities entertained by the classical mathematician. Anything short of this would constitute an unjustifiable limitation imposed on applied mathematics, whatever may be one's view of the limitations already imposed on pure mathematics.

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