Structuralism Without Structures

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1. Introduction: Approaches to Structuralism

As with many 'isms', 'structuralism' is rooted in some intuitive views or theses which are capable of being explicated and developed in a variety of distinct and apparently conflicting ways. One such way, the modal-structuralist approach, was partially articulated in my [1989] (hereinafter 'MWON'). That account, however, was incomplete in certain important respects bearing on the overall structuralist enterprise. In particular, it was left open how to treat generally some of the most important structures or spaces in mathematics, *e.g.*, metric spaces, topological spaces, differentiable manifolds, and so forth. This may have left the impression that such structures would have to be conceived as embedded in models of set theory, whose modal-structural interpretation depends on a rather bold conjecture, *e.g.*, the logical possibility of full models of the second-order ZF axioms. Furthermore, the presentation in MWON did not avail itself of certain technical machinery (developed by Boolos [1985] and Burgess, Hazen, and Lewis [1991]) which can be used to strengthen the program substantially. Indeed, these two aspects are closely interrelated; as will emerge, the machinery can be used to fill in the incompleteness so as to avoid dependence on models of set theory. The principal aim of this paper is to take the program forward by elaborating on these developments.

It will be helpful first, however, to remind ourselves of the main intuitive ideas underlying 'structuralism' and to indicate at least roughly where in the landscape of alternative approaches the one pursued here resides.

One intuitive thesis (one I explicitly highlighted in MWON) is this:

> Mathematics is the free exploration of structural possibilities, pursued by (more or less) rigorous deductive means.

Vague as this is, it already at least suggests the modern view of geometry, abstract algebra, number systems, and other 'abstract spaces', in which we attempt to characterize the structures of interest by laying down 'axioms' understood as 'defining conditions', which we may be able to show succeed

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in their role by producing a proof of their categoricity, and then proceeding to explore their (interesting) consequences. This in turn reveals the importance of second-order logical notions in mathematical foundations, for, as is well known, first-order renditions of defining conditions will inevitably fail to characterize certain of the most central structures in all of mathematics, including the natural-number structure, the reals, the complexes, and initial segments of the cumulative set-theoretic hierarchy. (Cf. Shapiro [1991]; also Mayberry [1994].)

A second intuitive principle, traceable to certain ideas of Dedekind [1888] and widely noted by philosophers and logicians, can be put thus:

> In mathematics, it is not particular objects which matter but rather certain 'structural' properties and relations, both within and among relevant totalities (domains).

To this one may wish to add:

> The very identity of individual mathematical objects depends on such structural relations (*i.e.*, on 'relative positions' in structures).

This is illustrated by pointing out that it is nonsensical, for example, to postulate a single real number (as Field [1980], p. 31, entertained; cf. Shapiro [forthcoming]); to be a real number is to be part of a complete, separable, ordered continuum. Particular constructions or definitions (*e.g.*, as convergent rational sequences) may, in given contexts, allow one to recover such structure, and, by focussing on a particular construction, it may appear that one could sensibly potulate a single such item (say, sequence); but we can only regard this as postulating a real number after we have recovered the structure, and so Shapiro's point stands.

Note that, in stating this second intuitive thesis, we have been intentionally vague about its scope. Is it understood as saying that *all* mathematical reference to objects is to be interpreted structurally (whatever that means precisely), or does it say, more modestly, that salient cases are? What of notions such as 'finite set' and 'finite sequence' of given objects? In contrast to numbers, the identity of a finite set of objects A, for example, seems determined by its members without considering its relative position within the naturally associated structure, the totality of finite A—sets ordered by inclusion, itself a fairly complicated infinitistic object. (Cf. Parsons's [1990] related points concerning 'quasi-concrete' mathematical objects.) And what of mathematical reference to structures themselves? Is there a regress involved in interpreting such reference structurally, and, if so, is it a vicious one? (Cf. Shapiro [forthcoming].) An adequate structuralism should somehow account for these apparent differences among mathematical concepts. As different approaches may be expected to treat such matters differently, let us take the second thesis in its limited, modest sense, allowing for supplementation as a particular view may require.

So understood, there are at least four main approaches to structuralism that have been proposed and should be distinguished:

(i) The framework of Model Theory ('MT'), carried out in Set Theory (say, ZFC). Structures are understood as models (sets as domains, together with distinguished relations and possibly individuals), and one can also speak of isomorphism types as 'structures' (at least one can if one is careful in either avoiding or admitting proper classes). (Of course, different choices of set theory yield different *explicanda*.) Equivalence of nominally distinct structures can be defined in terms of 'definitional extension' and related notions. (Thus, for example, the full second-order natural-number structure with just successor distingushed is equivalent to that obtained by adding addition and multiplication.)

(ii) The framework of Category Theory ('CT') (which itself can be axiomatized, as in Mac Lane [1986]). Structures are taken as the 'objects' of a category, treated as simples or 'points' by the axioms, and the 'morphisms' between the 'objects' typically preserve the characteristic 'structural properties' of the branch of mathematics in question. Thus, for example, isometries preserve metric structure, homeomorphisms preserve topological structure, diffeomorphisms preserve differentiable manifold structure, *etc.* Categories themselves can be treated as 'objects' in a category, and one can make sense of morphisms ('functors') preserving structural relations among the maps in the original categories. One even makes sense of morphisms ('natural transformations') of functors, giving rise to a functor category. (For an overview, see Mac Lane [1986], 386–406. For a categorial recovery of number theory, see McLarty [1993].)

(iii) Rather than realizing structuralism within an overarching existing mathematical theory, one may pursue a *sui generis* approach, taking structures to be patterns or universals in their own right. (See *e.g.*, Resnik [1981], Shapiro [1983], [1989], and [forthcoming]; for critical analysis, see Parsons [1990].) Different conceptions under this heading are possible, depending on the conception of universals. (See Shapiro [forthcoming].)

(iv) A modal-structural ('ms') approach, as in MWON. Here literal quantification over structures and mappings among them is eliminated in favor of sentences with modal operators. (Hence the term 'eliminative structuralism', see Parsons [1990], Shapiro [forthcoming]. And hence the title of this paper, which I owe to Shapiro.) The framework is a modal second-order logic with a restricted (extensional) comprehension scheme. (For details, see MWON, Ch. 1.) Categorical axioms of logical possibility of various types of structures replace ordinary existence axioms of MT or CT, and typical mathematical theorems are represented as modal universal conditionals asserting what would necessarily hold in any structure of the appropriate type that there might be. It turns out that a great deal of ordinary mathematics may thus be represented nominalistically, without the language of

classes at all, even under modality. (See Hellman [1994] and below.) Just how far this approach can be pushed is a somewhat open question, to be pursued further below. This brings us to the question already broached at the outset, whether this approach can do justice to 'structuralism' without a detour through (modal-structurally interpreted) set theory or category theory. In the following sections, we will present some evidence in favor of a positive answer.

Now this is not the place to undertake a systematic comparison of these alternative approaches. There are, however, two related contrasts of immediate concern between (i) through (iii) on the one hand and (iv) on the other that require our attention. The first pertains to the trade-off between platonist ontology and modality. The first three approaches are framed in modal-free languages but they are entangled well above the neck (naturally) in Plato's beard. Sets, categories, or universals are just taken as part of reality, leading to perennial disputation as to the nature of such 'things', how we can have knowledge of them or refer to them, *etc.*, and (of course) whether or not such questions are somehow misguided in the first place. Modal structuralism avoids commitment to such *abstracta*, at least in its initial stages (in treating, say, the number systems, prior to reconstructing set theory itself), and raises the prospect that a (modal) nominalistic framework may suffice to represent the bulk of ordinary mathematics.[1] (This depends on treating the second-order variables of the ms language nominalistically, but in ordinary contexts this can be done (see Hellman [1994] and below).) The price of course is taking a logical modality as primitive, raising questions of evidence and epistemic access not unlike those raised by platonist ontololgies. This trade-off is a subject of ongoing discussion, and will not be resolved here. We would point out, however, that assessing the trade-off depends on a better understanding of the alternatives themselves, including the ms approach. In particular, *just what modal-existence postulates are re-*

[1] The phrase 'ordinary mathematics' is not a precise one, but we intend it more broadly than do Friedman-Simpson *et al.* in the program of reverse mathematics, where explicitly excluded are 'those branches of mathematics which ... make essential use of the concepts and methods of abstract set theory', such as 'abstract functional analysis, general topology, or uncountable algebra'. (Brown and Simpson [1986], p. 123.) We do mean to exclude set theory and category theory themselves, but not the three fields just listed, nor the theory of non-separable Banach and Hilbert spaces, which by implication are also excluded by the Friedman-Simpson usage. The latter is motivated primarily by the question, 'What portions of ordinary mathematics can be carried out in which interesting subsystems of classical analysis (PA^2)?' and for this purpose, mathematical questions which cannot even be asked (even via suitable coding) in the language of PA^2 are sensibly excluded from 'ordinary mathematics'. Since we are under no such constraint, however, we can afford to be more liberal, counting as ordinary virtually any subfield short of those devoted to the grand foundational schemes. In our usage, certain 'concepts and methods of abstract set theory' can be deployed to some extent without commitment to abstract sets. But this can be spelled out without a precise use of the phrase 'ordinary mathematics'.

quired to implement structuralism? The more modest they are, the better the prospects for the ms approach. The results of our reflections below will bear directly on this. As will emerge, only rather modest modal-existence postulates are required; for much of mathematics, only countably many atoms need be postulated (as logically possible); for much more, including a great many topological structures and manifolds, uncountably many atoms are needed, but this need not transcend the scope of nominalism.

The second contrast between (i)–(iii) and (iv) concerns the wealth of mathematical structures incorporated within the respective framework. In the cases of (i)–(iii), the extent of richness is literally endless. While any particular set theory has its limitations, there are still boundless riches as regards the structures and spaces of ordinary mathematics. (In particular, there are no limits on cardinality or on type.) Category theory, especially with its large categories, is *prima facie* even more generous. And, presumably, *sui generis* universals are, as Quine might say, 'free for the thinking up'. Not that these frameworks avoid honest toil; nor that they are larcenous; they merely rely on the bountifulness of reality as they conceive it.

The case of the ms approach is more complex; indeed, we should distinguish two sub-approaches: (a) first develop a modal-structural interpretation of set theory (or of category theory), and then simply translate the MT (or CT) treatment of structures of interest accordingly; (b) seek a direct ms interpretation of theory of any such structures, avoiding set-theoretic commitments to whatever extent possible. From the perspective of ontology, it is (b) that is of greater interest. Moreover it confronts the challenge of describing interrelations of different types of structures, something that both MT and CT are set up to handle. If approach (b) were to be successful, structuralism would then stand independently of set theory rather than being just a chapter in it, even as interpreted; and it would represent a rather remarkable extension of nominalistic methods. It is this approach that we shall now continue to pursue.

(Traditionally, the problem with nominalism in mathematics has been not so much that Occam's razor has been dulled by Plato's beard, but rather that it has managed to remove the beard only by severing the head at the neck. Modal structuralism, as it has been extended (in my [1994] and below), manages a fairly clean shave while leaving the brain quite intact.)

2. Extending the Reach of Nominalism to Third-order Arithmetic and Third-order Analysis

The plan of this section is as follows. First we shall review the modal-structural frameworks for arithmetic and for real analysis developed in MWON, taking advantage of certain improvements since developed. These improvements consist principally in (1) the combined use (due to Burgess, Hazen, and Lewis [1991], henceforth 'BHL') of plural quantification (Boolos

[1985]) and mereology to *define* nominalistic ordered pairing in a general way (as opposed to adopting a new primitive pairing relation, as was suggested in MWON); and (2) the development of predicative foundations for arithmetic (in Feferman and Hellman [1995]) which enables modal-structuralism to get started, at least, in a manner compatible with predicativist principles. Having reviewed this, we will then indicate how the machinery just referred to under (1) can be used to extend the reach of nominalism one level beyond each of the core systems of arithmetic and real analysis described in MWON–essentially how to pass from PA^2 to PA^3 and from RA^2 to RA^3. Some of the benefits of these extensions regarding structuralism will then be explained in the next section.

Beginning with the standard Peano-Dedekind axioms for the natural numbers, PA^2, involving just successor, $'$, and the second-order statement of mathematical induction, we treat an arbitrary sentence S of first- or second-order arithmetic (in which any function constants have been eliminated by means of definitions in terms of $'$) as elliptical for the modal conditional

$$\Box \forall X \forall f [\wedge PA^2 \rightarrow S]^X ('/f),$$

in which a unary function variable f replaces $'$ throughout and the superscript X indicates relativization of all quantifiers to the domain X. This is a direct, modal, second-order statement to the effect that 'S holds in any model of PA^2 there might be'. (Note that use of model-theoretic *satisfaction* is avoided.) This was called the 'hypothetical component' of the modal-structural interpretation (msi) of arithmetic. In order that this provide a faithful representation of classical arithmetic, it is also necessary to add a 'categorical component', a statement that such structures (ω-sequences or $\mathbb{N}$-structures) are logically possible:

$$\Diamond \exists X \exists f [\wedge PA^2]^X ('/f). \qquad \text{(Poss } \mathbb{N})$$

This is the characteristic modal-existence (mathematical existence) claim underlying classical arithmetic (or 'classical analysis', logicians' term for PA^2). It distnguishes the modal-structural approach from 'deductivism' and from 'if-thenism'. All sentences of the original mathematical language (for PA^2) are regarded as truth-determinate regardless of their formal provability or refutability. Furthermore, various arguments show that the translation scheme respects classical truth-values. A key step is the recovery of Dedekind's categoricity proof, that any pair of models of PA^2 are isomorphic. This can be carried out within modal second-order logic, using just the ordinary second-order (extensional) comprehension scheme (with ordinary universal quantifiers in the prefix, not boxed ones) and basic quantified modal logic (although S-5 is the preferred background). Here the reasoning is straightforward mathematical reasoning under the assumption that a pair of PA^2 structures is given. Appeal to intensions—relations across possible worlds, as it were—can be avoided if we assume an 'accumulation principle' to the effect that if it is possible there is an ω-sequence with (PA^2-definable) property P (which, by quantifier relativization, involves only items internal to the given sequence) and it is possible there is another ω-sequence with such property Q (internal to its sequence), then the conjunction of these existential statements is also possible, *i.e.*, these two sorts of ω-sequences occur in the same world, so to speak. (For further details, see MWON, Ch. 1; also Hellman [1990].)

Note that talk of 'possible worlds' is heuristic only; the modal operators are primitive in the framework and are not required to be given a set-theoretical semantics. Note further that the accumulation principle derives its plausibility from the combination of two considerations: first, it is only logical possibility that is at issue, and second, the mathematical properties labelled 'P' and 'Q' are entirely 'internal' to their respective sequences, as relativization to the respective domains of any quantifiers they may contain insures. The essential point is that anything internal to a given structure cannot conflict with anything internal to another, so that structures satisfying the respective conditions are logically compossible. Thus, there is no requirement that the structures involved be of the same general type. One could be an ω-sequence and another could be an ordered continuum, or whatever. Moreover, the principle can be generalized in the obvious way to cover any finite number of structures. These generalizations are important for this approach to structuralism, since, as in set theory or category theory, we often wish to speak of relations among a variety of structures. Finally, note that the formulas above quantify over structures by quantifying directly over their domains and distinguished relations or functions; it is not necessary to ascend a further level in type as is commonly done in model theory. From the rest of the formula, it can always be made clear which relations or functions are defined on which domains.

So far we have used some of the ordinary language of mathematics—the language of 'domains' and 'functions'—to eliminate reference to 'numbers' as special objects. Arithmetic is not about special objects; it is rather about a special type of structure. In accordance with part of Dedekind's conception, it investigates facts which hold of any 'simply infinite system', but we have explicitly used modal operators both to get away from commitments to any special instantiation of the structure-type and to achieve an open-ended generality appropriate to mathematics.[2] On this conception,

[2] As Tait [1986] and Parsons [1990] have pointed out, it would be a mistake to attribute to Dedekind himself an eliminativist structuralist position (cf. MWON, Ch. 1). In the text, however, we are referring to Dedekind's conception of 'the science of arithmetic' as investigating what holds in any simply infinite system. This, surely, is the starting point of any eliminativist approach.

mathematics investigates a certain category of necessary *truths*, not confined to what happens to exist. But it does not have to postulate a special realm of necessary *existents* in the process. Only one level of *abstracta* has been invoked, corresponding to the second-order variables. And, it should be noted, the second-order comprehension scheme does *prima facie* commit us to actual classes and relations of whatever actual first-order objects we recognize among the *relata* of the relations (or relation variables) of our language. Significantly, there is no iteration of collecting, so this is not Plato's full beard, to be sure; but it does seem more than just a six o'clock shadow! So how do we get a clean shave?

Well, it turns out, there are many ways to shave a beard, at least at the stage of second-order arithmetic. One way is that of *predicative foundations*, in the tradition of Poincaré, Weyl, and Feferman *et al.* The central idea here is to restrict comprehension axioms to *definable* classes (and relations), where this is spelled out in terms of formulas of mathematical language whose quantifiers range over already defined or specified objects. (For details on various options, including systems of variable type, see writings of Feferman, *e.g.*, [1964], [1968], [1977], [1988].) Typically one begins by taking the natural numbers for granted and considering, first, those sets of natural numbers definable by arithmetic formulas (with quantifiers only over natural numbers)—the first-order sets—and then sets of natural numbers definable by formulas with quantifiers over numbers and first-order sets, and so on. (How far this may be iterated is a delicate matter.) This can qualify as nominalistic—relative to the natural numbers—in that one can eliminate reference to sets and relations in favor of the semantic notion of *satisfaction* of formulas by natural numbers, or by other nominalistically acceptable objects, *e.g.*, predicates themselves. (Cf., *e.g.*, Chihara [1973], Burgess [1983].)

The problem with this as a nominalization program, however, is that the natural-number structure has been taken as given. And this has appeared unavoidable, for there is *prima facie* reliance on *impredicative* class existence principles in the classical constructions of the natural-number structure, *e.g.*, the Dedekind-Frege-Russell definition as (essentially) the intersection of all inductive classes containing 1 (or 0). Such principles are standardly used also to prove the existence of an isomorphism between any structures satisfying the PA^2 axioms; and it is well known that the second-order statement of induction is necessary for this result. Contrary to these appearances, however, the natural-number structure can itself be constructed predicatively, beginning with the notion of *finite set* governed by axioms that are intuitively evident or of a stipulative character, as carried out by Feferman and Hellman [1995]. Of particular interest here are the facts that mathematical induction is itself derivable from within an elementary theory of finite sets and classes ('EFSC'),[3] and that Dedekind's categoricity proof ('*unicity* of the natural number structure') is also recoverable. (Cf. Feferman-Hellman [1995].)

This still leaves us with the non-nominalist notion of 'finite set' governed by the EFSC axioms. But even this vestige of platonist commitment (5 o'clock shadow?) can be eliminated. One can first postulate the logical possibility of an infinitude of atoms (atomic individuals, governed by the axioms of atomic mereology; cf. Goodman [1977], also MWON, Ch. 1), and then interpret 'finite set' as 'finite sum (or whole, or fusion) of atoms'. To express 'infinitude of atoms' one can use the device of plural quantification and postulate:

'There are (possibly) some individuals one of which is an atom and each one of which fused with a unique atom not overlapping that individual is also one of them.' (Ax ∞)

With this postulate, one has the essentials of a mereological model of the EFSC axioms: first-order variables can be taken to range over arbitrary individuals (atoms and fusions of atoms); finite set variables range over finite fusions of atoms; class variables range over arbitrary fusions of atoms. If a null individual is admitted (as a convenience), the axioms are satisfied as they stand; otherwise the comprehension axioms can be complicated slightly to avoid the null individual. (EFSC also takes a pairing function as primitive, governed by two axioms: P–I, the standard identity condition, $(x, y) = (u, v)$ iff $x = u$ and $y = v$, and P–II, existence of an urelement under pairing. It turns out that both these are satisfied on, say, the Burgess construction of nominalistic pairing in BHL [1990]. So we can invoke this in interpreting pairing in EFSC.) Now, within such a model of EFSC there is a mereological model of the PA^2 axioms. (This follows from the construction of an $\mathbb{N}$-structure in EFSC.) Moreover, as the class variables are taken to range over arbitrary individuals (fusions of atoms), we even have a *full* second-order PA^2 model in the classical sense, in which arbitrary sets of numbers correspond to arbitrary fusions of the individuals serving as numbers. The predicativist may stop short of this, confining oneself to 'definable fusions' of atoms in specifying the range of the class variables. (Cf. Hellman [1994], sec. 3.) But the essential point here is that both predicativist and full second-order arithmetic are interpreted nominalistically.

[3] The system EFSC is formulated in a three-sorted language with individual variables, variables for finite sets of individuals, and variables for classes of individuals. Formulas with no bound class variables are called 'WS' formulas (for 'weak second-order'). A pairing operation-symbol is primitive as is $\in$ relating individuals to finite sets and classes. The logic is classical (with equality in the first sort). The axioms of EFSC are, in words, as follows:

(WS-CA)	Weak second-order comprehension: existence of classes as extensions of WS-formulas;
(Sep)	Separation for finite sets: existence of a finite set as the intersection of any given finite set and the extension of a WS formula;
(Empty)	Existence of the empty finite set;
(Adjuction)	Existence of a finite set obtained by adjoining any single individal to a given finite set;
(Pairing I)	'Pairs are distinct just in a case either first or second members are';
(Pairing II)	Existence of urelements under pairing.

The system EFSC* is obtained from EFSC by adding the axiom,

(Card) 'Any finite set is Dedekind-finite.'

In Feferman-Hellman [1995], the existence of structures satisfying mathematical induction is derived in EFSC*; however, as Peter Aczel has pointed out, this can already be proved in EFSC.

From here, one could continue on with predicativist analysis, constructing countable analogues of the classical continuum, made up of 'definable' or 'specifiable' reals, which support rather rich portions of functional analysis and related subjects. (Cf., *e.g.*, Feferman [1988].) As Feferman has pointed out, within various systems of predicative analysis, one can even prove the unicity of the real number structure, as seen from within that system: impredicativity is avoided because one requires, not the full classical principle of Continuity (least upper bound axiom), but only the sequential form, 'Every non-empty bounded sequence of reals has a least upper bound'. (For details, see Hellman [1994], sec. 2.)

Continuing along the predicativist route, one can introduce symbolism for reasoning about (specifiable) classes (and relations and functions) of reals, classes (*etc.*) of classes of reals, and so on through the finite types. (See, *e.g.*, Feferman's system W [1988] and [1992].) At each level, one is considering not the full classical totalities, of ever higher uncountable cardinality, but subtotalities of objects predicatively specifiable in mathematical language.[4] Thus, the ranges of the quantifiers at each level are really countable, although from within the predicativist system they may be described as 'uncountable'. (The predicativist can carry out the reasoning of Cantor's diagonal argument, but, implicitly, only *specifiable* enumerations are considered.)

To what extent can predicativism carry out a structuralist program for mathematics? This is a large question which cannot be fully answered here. But the following points towards an answer may be offered. First, one must be more precise about what it means to 'carry out a structuralist program'. Presumably this includes these things: (i) characterizing the types of structures or spaces that arise in the various branches of mathematics (or at least 'ordinary mathematics' as we have used that term above); (ii) describing the main types of relationships among these structures, including the various morphisms within and among them (isomorphisms, homomorphisms, embeddings of various sorts, *etc.*); (iii) recovering the important theorems concerning the various structures and relations among them, including existence theorems.

[4] For ways of making this precise in connection with unramified systems such as W, see my [1994], n. 2.

Judged by these standards, predicativism gets mixed reviews. It does surprisingly well, for example, in recovering theories of various types of metric spaces central to scientifically applicable mathematics. Although the concept of Lebesgue outer measure is not predicatively available, theories of measurable sets and measurable functions can be developed (cf. Feferman [1977], §3.2.5), and then one can obtain the L_p spaces and carry out a structuralist treatment of Banach and Hilbert spaces. On the other hand, clearly there are important limitations: 1) Various objects of importance in the classical (set-theoretic) treatments are simply not available, *e.g.*, outer measures, as just indicated, and general descriptions of major types of spaces are *prima facie* impredicative, *e.g.*, topological spaces with families of open sets closed under *arbitrary* unions. 2) Proofs of key theorems, even if predicatively statable, may require impredicative constructions essentially. A known example is Friedman's finite form of Kruskal's theorem on embeddability of finite trees. (See *e.g.*, Smorynksi [1982].) Even if this example does not pertain to scientifically applicable mathematics, it surely pertains to significant mathematical structures. 3) Even in the cases in which predicative proofs of key theorems are possible, *e.g.*, the unicity of the real-number structure, in reality we know—as does the predicativist—that the structures to which the result pertains are countable, hence only small parts of the structures classically conceived. Even if the predicativist makes no direct sense of the latter phrase, one can pass to more encompassing, predicatively graspable totalities, essentially by enriching a given language with predicatively intelligible semantic machinery for defining new objects, *e.g.*, real numbers. (Cf. Hellman [1994], §2.) If one is a skeptic about uncountable totalities generally, then presumably one is willing to pay the price of this language-relativity of much of mathematics. If, however, one follows the classicist in taking seriously the absoluteness of uncountability—*e.g.*, if one treats totalities such as 'all sets of natural numbers' or 'all fusions of countably many atoms' as having a definite and maximal sense—then one will regard the predicativist substitutes as falling far short of the genuine articles.

This much should be clear: If the objection to the uncountable is motivated by nominalist concerns—the desire to avoid commitment to classes or universals, *etc.*—then it is misplaced. A fusion of atoms is just as 'concrete'—just as much not a class or a universal, *etc.*—as the atoms themselves. In the language of types, both are of type 0. It does not matter how many there are. If we are given countably infinitely many atoms—by definition pairwise discrete—then we may speak of arbitrary fusions of them without nominalistic qualms. Then to go on to say how many of such fusions there are requires further reasoning, to be sure; but in fact there is no problem in carrying out Cantor's diagonal argument nominalistically to convince oneself that there are uncountably many. And if in the course

of this reasoning, one meant to consider 'any possible enumeration'—not merely any that could be specified in some privileged symbolism as the predicativist intends—then the conclusion has its absolute force.

This leads us to relax the 'definitionist' stance of predicativism in pursuit of a nominalist structuralism. In our nominalist Σ-comprehension scheme,

$$\exists x\Phi(x) \rightarrow \exists u\forall y[y \circ u \leftrightarrow \exists z(\Phi(z) \ \& \ z \circ y)] \qquad (C\Sigma)$$

(in which $\circ$ is 'overlaps' or 'contains a common part with'), we allow the predicate Φ (lacking free u) to contain quantifiers over *arbitrary* individuals, whether or not specifiable by any particular symbolic means. An immediate consequence is that, once we have postulated (Ax ∞)—guaranteeing an ω-sequence of atoms (Poss $\mathbb{N}$)—we already have embedded within such a sequence enough subsequences to serve as arbitrary real numbers. Standard arithmetization procedures can be used to introduce negative integers, rationals, and then reals (either as Cauchy sequences of rationals or as Dedekind cuts). (By the device of numerical pairing, that is pairing *via* the atoms of the postulated ω-sequence, one can remain within that structure; reals are then just certain fusions of atoms.) The following important facts should be noted:

(1) The full classical Continuity principle (lub principle for arbitrary nonempty bounded sets of reals) is derivable along logicist lines (using CΣ), without exceeding the bounds of nominalism. (The usual set-theoretical arguments are avialable making use of plural quantifiers to get the effect of quantification over sets and functions of reals.)

(2) The categoricity of real analysis (RA) is also derivable within this nominalist framework, in the sense that any two concrete $\mathbb{R}$-structures (*i.e.*, with reals built up from concrete ω-sequences in logicist fashion as just alluded to, together with the usual ordering $<$ on reals) are isomorphic. (The proof of this requires even less than (1), *viz.* Sequential Completeness rather than the full lub principle suffices. For a visualizable nominalistic construction, see Hellman [1994].) Furthermore, since all fusions of atoms count as individuals, regardless of specifiability by formulas, this categoricity proof has the absolute significance of the standard set-theoretical one.

Thus, (Poss $\mathbb{N}$)—hence (Ax ∞)—suffices for a nominalist structuralist treatment of full classical analysis (PA^2). In particular, the criteria (i)–(iii) above are met with respect to PA^2-structures. But, as the alert reader may have noticed, we have enough machinery at our disposal to ascend one more level, to third-order number theory (PA^3). Plural quantifiers achieve the effect of quantification over sets of reals, as just described; and the device of BHL pairing reduces polyadic quantification at this level (over relations of reals) to monadic. Still we have only had to hypothesize a countable infinity of atoms.

There may be the concern that plural quantifiers (at a given level, say pluralities of reals) do not really get around classes (of reals); that a sentence such as, 'Any reals that are all less than or equal to some real are all less than or equal to a least such', really concerns classes of reals as values of a hidden variable. (No first-order conditions determine the same class of models.) I agree with Boolos [1985] and others, however, that we do have an independent grasp of plural quantifiers and their accompanying constructions, and that we can use them to formulate many truths at any given level that would have to be regarded as false on an ontology that repudiates classes of objects at that level. (The case based on this and related points has been made very effectively by David Lewis in his [1991], §3.2.) The school teacher who says, 'I've got some boys in my class who congregate only with each other', should not be ascribed on that basis a commitment to classes other than school classes. The EPA official who says, 'Some cars are tied with one another in being the most polluting vehicles on the road', need not be committed to classes of vehicles other than the usual predicative ones (two-door sedan, station wagon, *etc.*). And the nominalist who entertains just some atoms and their fusions can even go on to say things like, 'Some of those fusions can be matched up in a one-one manner with the atoms whereas not all of them can', without implying anything about any objects of higher type than the fusions themselves. (In virtue of BHL pairing, even the talk of one-one correspondences is innocent.)

Thus the strength of full, classical third-order number theory—equivalently, second-order real analysis—is attained within a nominalist modal-structural system without postulating more than a countable infinity of atoms. This is already quite a rich framework for carrying out structuralism. But if we are prepared to entertain at least the logical possibility of a continuum of atoms, we can attain one full level more, that of *fourth-order number theory*, equivalently *third-order real analysis*. Suppose we postulate the possibility of a complete, separable ordered continuum ('$\mathbb{R}$-structure', for short) of atoms, which we may write

$$\diamond \exists X \exists f [\wedge RA^2]^X (< / f), \qquad \text{(Poss } \mathbb{R}\text{)}$$

in which RA^2 denotes the axioms for such a structure (with first-order quantifiers stipulated to range over atoms), including the second-order statement of Continuity, and where a relation variable f replaces the ordering relation constant $<$ throughout; then our second-order variables already range over arbitrary fusions of these, at the level of sets of reals. Functions and relations of reals are reducible to sets *via* pairing. Quantifying plurally over sets of reals then gives the effect of quantifying (singularly) over sets of sets of reals and, *via* BHL pairing, over functions and relations of sets of reals (hence also of functions and relations of reals). This is mathematically at the level of RA^3 or PA^4, a very rich framework indeed. (Note that, since fusions of (atoms serving as) reals are entertained as *bona fide* objects, it is pluralities of them that we plurally quantify over to achieve the next level. Even if plural quantification is already invoked lower down, to introduce pairs, triples, *etc.* of reals so as to reduce relations of reals to sets, we get the effect of plurally quantifying over relations of reals when we plurally quantify over sets of reals. We are not 'plurally quantifying over pluralities' except in this innocent sense.)

In order to characterize $\mathbb{R}$-structures, it is necessary that we use the second-order statement of the Continuity principle (lub axiom); it is one of the defining conditions built into the modal-existence postulate (Poss $\mathbb{R}$). Does this mean that we have given up the advantages of a logicist-style *derivation* of Continuity from more elementary, general principles? Well, yes—but also no! Yes, in that the principle (or a geometric equivalent) is essential in asserting the possibility of an $\mathbb{R}$-structure *whose first-order objects are atoms.* But no, in that we have already derived Continuity in logicist fashion (but nominalistically) above as it governs $\mathbb{R}$-structures built from fusions of atoms of an $\mathbb{N}$-structure. If reals are taken as constructed objects (*e.g.*, Dedekind sections, *etc.*), then we already have (Poss $\mathbb{R}$) based on (Poss $\mathbb{N}$), as already described. (This point was not made in MWON, for, not utilizing the resources of plural quantifiers, we did not have the means to state Continuity in full generality, nominalistically, given just (Poss $\mathbb{N}$). A major advantage of the present approach is that, now, we do have the means to speak of arbitrary pluralities of reals, given just (Poss $\mathbb{N}$).) We give none of this up when we *add* the postulate of an *atomic* $\mathbb{R}$-structure. But of course we are adding something substantial. In terms of the familiar aphorism, there are two cakes: we are having one while eating the other.

Now the worry arises that in entertaining anything so idealized and remote from experience as spatio-temporal or geometric points (typically thought of as realizing the RA axioms) we have exceeded the bounds of nominalism. Surely a case can be made that 'points of space-time'—especially 'unoccupied points'—are in some sense 'abstract', even if invoked by standard formulations of physical theory. Field [1980], who based his nominalization program on the acceptability of space-time points and regions, sought to answer such objections, in part by appealing to substantivalist interpretations of space-time physics which do seem to accord a kind of causal role to (even unoccupied) space-time (*e.g.*, as a field source with clearly physical effects). And certainly this neo-Newtonian perspective is not without its adherents. But it is important to realize that the modal-structural axiom, (Poss $\mathbb{R}$), does not ultimately depend on a substantivalist view of space-time. For, just as the logical modality frees us from having to posit actual countable infinities, so it frees us from any particular view of actual space, time, or space-time. It suffices, for example, that Newtonians who posited a continuous luminiferous medium were entertaining a logically coherent possibility. Similarly one can coherently imagine a perfect fluid filling, say, a bounded open region and satisfying whatever causal conditions one likes. All that matters is that it be conceived as made up of parts satisfying the RA^2 axioms; it need not involve 'unoccupied space'; it need not be useful to any explanatory purpose; it need not be (period). Furthermore, supposing that such possibilities are 'world-independent'—that they have the absolute status generally accorded logical possibilities—frees mathematics based on (Poss $\mathbb{R}$) of any contingency. (In the modal system S-5, appropriate to logical possibility, we have Nec Poss $\mathbb{R}$ once we have Poss $\mathbb{R}$.) There is no 'lucky accident' involved in the truth of modal-structuralist mathematics.

Since the comparison with Field's program has come up, it is appropriate to emphasize that, for Field also, there is no 'lucky accident' involved in the truth of mathematics, for (except in a vacuous sense) Field's program is instrumentalist, not recognizing mathematical truth at all, except in vacuously assigning 'false' to all existential statements and 'true' to anything equivalent to the negation of one, since 'there are no numbers'! This is as great a contrast with modal-structuralism—nominalist or not—as there could be.

Yet perhaps the contrast is at bottom illusory. For, in seeking to recover nominalistically various arguments that certain mathematical-physical theories are semantically conservative with respect to their nominalized counterparts, Field [1989] [1992] introduced logical modality as a means of bypassing reference to models. Along the way he seems committed to affirming such things we would write as,

$$\diamond \wedge PA^2, \qquad \text{(FC)}$$

that the conjunction of the second-order Peano-Dedekind axioms is logically possible. ('FC' is for 'Field's commitment'.) Now this is not quite the same as our (Poss $\mathbb{N}$), which involves domain and function variables, but (Poss $\mathbb{N}$) follows in second-order logic from (FC). Mathematically speaking, they are not essentially different. (Philosophically they are. As explained in MWON, Ch. 1, problems arise in interpreting the function constant of (FC) under the modal operator, problems that (Poss $\mathbb{N}$) avoids. And explicit reference to domains is natural in a structuralist interpretation.) But then *everything needed for a modal-structuralist treatment of mathematics in PA^2—or even PA^3, if plural quantifiers are added—is available. So Field's system is not really instrumentalist after all.* It embraces the core of the ms interpretation but then does not go on to utilize it. In this sense, much genuine mathematics is still present.[5]

[5] Concerning Field's commitment (FC), let 'Ax' stand for the conjunction of axioms

3. Realizing Structuralism

Without attempting anything like a survey of mathematical structures, let us indicate by salient examples how the above described frameworks suffice for a great deal of the structuralist enterprise. Let us refer to those frameworks as PA^3 (or RA^2) and PA^4 (or RA^3), understanding of course that it is the nominalistic modal-structural version of these systems that is meant, that is, the relevant modal-existence postulate, *e.g.*, (Ax ∞), with the background of S-5 modal logic, mereology with the comprehension scheme (CΣ) for modal-free formulas, plural quantifiers, and the assumptions needed for BHL pairing (as in the Appendix of Lewis [1991]). (The second-order logical notation of the modal-existence postulates may be retained, understanding the monadic second-order variables to range over arbitrary fusions of atoms and polyadic second-order variables to range over fusions of n-tuples of individuals based, say, on b-pairing of BHL.) It should also be understood that, in treating many abstract structures, what we really require of these frameworks is not the full structure of an $\mathbb{N}$-structure or an $\mathbb{R}$-structure, but merely a domain with sufficiently many atoms, *e.g.*, a countable infinity or continuum many. Then one needs the specific functions, relations, or other items characteristic of the type of abstract structure in question, *e.g.*, a metric, a topology, a chart-system, *etc.* Thus when we appeal to PA^2 (or PA^3), say, we may only be appealing to the modal-existence of a countable infinity of atoms; and when we appeal to RA^2 (or RA^3), we may only be appealing to the modal-existence of continuum-many atoms. Alternatively, we may only wish to appeal to the modal-existence of uncountably many atoms, leaving open the cardinality. It should be clear how to formulate this using the available machinery, mereology, plural quantifiers, and BHL pairing. We need merely translate the statement,

$\diamond \exists X$['X is a fusion of infinitely many atoms & a proper part Y of X is

of a finitely axiomatized mathematical theory T (such as PA^2), and let S be a nominalistically formulated statement (of applied mathematics); then Field's modal formulation of the conservativeness of T is

$$\text{If } \diamond S, \text{ then } \diamond (Ax \ \& \ S), \qquad \text{(Conserv } T)$$

where Field's $\diamond$ operator is said to mean 'it is not logically false that'. (Cf. Field [1992], p. 112.) Initially, Field focuses on first-order theories (for which (Conserv T) may be a *schema* with finite conjunctions of axioms), but later he considers some second-order cases as 'not completely without appeal'. Moreover, since the 'complete logic of Goodmanian sums' is invoked (in Field [1980]) in connection with representation theorems for $\mathbb{R}^4$—*i.e.*, even arbitrary fusions of space-time points are recognized—there should certainly be no objection to (Conserv PA^2). (Indeed, as we have seen, Ax in this case is nominalistically interpretable without anything so strong as space-time substantivalism. Field's reservations in connection with second-order principles seem to have primarily to do with transfinite set theory and not with arithmetic or analysis. Cf. his [1992], p. 119.) But of course there are some S such that Field accepts $\diamond S$, and then $\diamond Ax$ follows.

It may well be that Field's informal understanding of the logical modality differs in some respects from that expressed in MWON, but his core modal logic, like that of MWON, is a version of S-5, and it seems clear that a great deal of modal mathematics (of PA^2 and even RA^2) can be carried out within Field's system.

a fusion of countably many atoms (which we can say directly by saying Y is in one-one correspondence with any infinite part of Y) & X is not in one-one correspondence with Y'].

If we wished to specify a fusion of $\aleph_1$ atoms, we could simply add to this that X is in one-one correspondence with any infinite part which is not in one-one correspondence with Y. Similarly we could specify a fusion of $\aleph_2$ atoms, and so on. (Writing such things out in primitive notation with all the plural quantifiers needed for BHL pairing would not be pleasant, but that is why we like quotes. For more on the expressibility of cardinality, see Shapiro [1991].)

Let us begin with the important example of *metric spaces*. Abstractly considered, these are pairs consisting of a domain X and a function d (positive definite metric) from $X \times X$ into $\mathbb{R}$ satisfying the familiar first-order conditions: $d(x,x) = 0$, $x \neq y \rightarrow d(x,y) > 0$, $d(x,y) = d(y,x)$, and $d(x,y) + d(y,z) \geq d(x,z)$ (triangle inequality). Bijective mappings ϕ between two such spaces, (X,d) and (X',d'), preserving metrical relations are called *isometries*. (That is, $d(x,y) = d'(\phi(x),\phi(y))$.) Even if we require that the domains be uncountable, all this is describable at the level of PA^3: the domains are at the level of sets of reals, metrics are at the level of sets of ordered triples of reals, hence sets of reals *via* pairing, and isometries are at the level of sets of ordered pairs of reals, hence sets of reals. Thus we can quantify over structures of this type and isometries (and similar relations) among them without positing more than a countable infinity of atoms.

Of course, many metric spaces of importance carry additional structure, often embedded in the elements of the spaces themselves. For example, Banach spaces are normed linear spaces whose elements are often real- or complex-valued functions; they become metric spaces under the definition $d(f,g) = \|f - g\|$. Hilbert spaces have the additional structure of an inner product, (,), which gives rise to a norm *via* $\|f\|^2 = (f,f)$. Such spaces are called *separable* if they include a countable dense set of vectors (where density means that any f in the space can be approximated arbitrarily closely in the metric by elements of the countable subset). The vectors of a separable Hilbert space are codable as real numbers, and inner products are then at the level of sets of reals (*via* pairing). Linear operators are also at this level, as are subspaces. The latter, however, can be coded as reals since a subspace is spanned by countably many vectors. The same is true for a great many operators, *e.g.*, the continuous (= bounded) ones. Norm-preserving maps, *unitary transformations*, are also at the level of sets of reals. Thus, the metrical structure of separable Hilbert spaces is describable in the PA^3 framework, again not exceeding the postulation of countably many atoms.[6] Significantly, the classical demonstration that the axioms for separable Hilbert spaces (of infinite dimension) are categorical carries over intact. (The standard examples ℓ^2 and L^2 are available.) Indeed, even measure-theoretic structure can be captured at this level. Note, finally, that—in virtue of the unrestricted second-order (extensional) comprehension principles of our background logic—theorems of the standard classical theories of these structures translate into theorems of the modal-structuralist framework, as in the cases of arithmetic and analysis. (Cf. MWON, Ch. 1.) This preservation of theorems holds regarding all the structures we shall consider, so we need not keep repeating the point.

For some theorems involving multiple structures, instances of accumulation principles, described above, may be used. It should be mentioned, however, that in many cases, appeal to such principles can be bypassed and one can reproduce constructions and theorems regarding multiple structures—indeed even infinite classes of structures—by considering certain parts of a single possible universe of infinitely many atoms, each such part endowed with relevant mathematical structure. The example of product spaces in topology is a good illustration and will be described briefly below. (Whether the class of structures can be uncountably infinite depends on the case.)

Let us consider now some structures from *measure theory*, which has posed a challenge to various constructive programs. (We follow Halmos [1974].) Central are *measure spaces*, triples (X, S, μ) where X is a domain of points (*e.g.*, real numbers), S is a class of (measurable) subsets of X (a σ-ring) whose union is X, and μ is an extended real-valued, non-negative, countably additive set function on S assigning 0 to the empty set. Now, if S is the σ-ring generated by a collection of cardinality of $\mathbb{R}$ (*e.g.*, S is the collection of Borel sets of reals, generated by the bounded left semi-closed intervals), it has cardinality of $\mathbb{R}$ also. Then the members of S can be coded as reals so that μ is of the type of a function from $\mathbb{R}$ to $\mathbb{R}$, at the level of PA^3, and so within the framework of ms arithmetic.

Beyond this, however, one quickly encounters measure-theoretic structures that require resources beyond PA^3. Let X be the real line, $\mathbb{R}^1$, S the class of Borel sets, and μ Lebesgue measure on S. Let $\bar{S}$ be the result of adding to S all sets of the form $E \cup N$ where $E \in S$ and N is a subset of a member of S of measure 0, and let $\bar{\mu}$ be the completion of μ on $\bar{S}$ ($\bar{\mu}(E \triangle N) = \mu(E)$, E and N as just described, where $\triangle$ is symmetric difference). $\bar{\mu}$ is the complete Lebesgue measure, and the sets of $\bar{S}$ are the Lebesgue measurable sets. Now, since there are uncountable (Borel) sets of (μ) measure 0 (of cardinality of the continuum, in fact, by Cantor's

[6] Indeed, if the vectors of the dense set are codable as natural numbers, as they are in standard examples such as the ℓ^p spaces, then the metrical structure of the separable Hilbert space is decribable in PA^2, and indeed much of the theory can be developed in very weak subsystems. (See Brown and Simpson [1986].) However, the completion of the countable dense substructure does not formally exist in PA^2; a general structuralist treatment naturally will distinguish the uncountable completions from their countable codes, and this requires ascent past PA^2.

middle third construction), and since $E \in S$, $\mu(E) = 0$, & $F \subset E$ together imply $F \in \bar{S}$, the cardinality of $\bar{S}$ must be that of the power set of $\mathbb{R}$ (by Cantor's cardinality theorem). (Cf. Halmos [1974], p. 65 (5), (6).) Thus $\bar{S}$ and $\bar{\mu}$ are essentially at the level of classes of sets of reals, *i.e.*, at RA^3, one step beyond PA^3. But RA^3 suffices; this structure is describable within (nominalistic) ms analysis, and its modal-existence assured.

Similar remarks apply to the theory of outer measure. The relevant structures are extensions of a measure space, (X, S, μ), of the form $(X, H(S), \mu^*)$, where $H(S)$, the *hereditary σ-ring generated by S*, is the smallest σ-ring containing the sets of S and closed with respect to (arbitrary) subsets of such sets, and μ^* is the outer measure defined on $H(S)$ by

$$\mu^*(E) = \inf\{\mu(F) : E \subset F \in S\}.$$

Here again both the class of (outer) measurable sets and the (outer) measure are at the level of classes of sets of reals, hence at RA^3 (PA^4). Once again, the resources of this framework are needed but suffice for this type of structures.

Turning now briefly to topological spaces, the story is similar. In general, these are structures of the form $(X, \mathcal{O})$, where X is a set and $\mathcal{O}$ is a class of *open subsets* of X such that $\mathcal{O}$ contains the empty set and X and is closed under finite intersections and *arbitrary* unions. This leaves open a vast array of possibilities. At one extreme, the *trivial* topology contains just the empty set and X, and at the other extreme the *discrete* topology contains every subset of X. In between lie the most familiar topologies of the real line, the plane, *etc.* (the usual topologies of $\mathbb{R}^n$), which are *separable*, *i.e.*, have a countable base (*e.g.*, the open n-spheres of rational radii). This means that for every point x in an open set U there is a basic open set B such that $x \in B \subset U$. Thus, every open set in such a topology can be represented as a countable union of basic open sets. In the case of $\mathbb{R}^n$, this means that every open set can be represented by a real number, so that $\mathcal{O}$ itself can be represented as a set of reals. Thus, such topological spaces can be described in PA^3, modal-structural arithmetic. (Similarly for the various spaces encountered in point-set topology.) However, this representation relies on metrical information which is generally not available. A long story here can be shortened considerably by noting that the discrete topology (on X) is a worst case, concerning cardinality (which is the chief guide in determining type level). Thus if X has cardinality of the continuum, any topology on X has cardinality no greater than the power set of the continuum, *i.e.*, $\mathcal{O}$ is at or below the level of a class of sets of reals, at RA^3. And morphisms between such spaces (especially homeomorphisms) are generally capturable at RA^2. Thus, a very rich variety of topological spaces is certainly describable within the nominalist ms framework. Even if higher set theory (even Morse-Kelley) is needed for a completely general theory of topological spaces (free of any cardinality restrictions whatever), clearly a great deal of the subject can be developed on the basis of our two modal-existence postulates above, that is without officially countenancing sets or classes at all. (Note here that the predicativist alternative does not extend nearly so far, as the very notion of closure of a topology under arbitrary unions is not available. This serves further to highlight the contrast between nominalism and even liberal varieties of constructivism.)

Let us end this cursory tour by mentioning three examples of topological spaces of some complexity, as well as importance, that can be treated within our framework. One example is that of n-dimensional manifolds, of great importance in space-time physics. As was already described in MWON, Ch. 3, the general theory of such structures does exceed the reach of RA^2, but only by one level. The effect of adding plural quantifiers is that we now have RA^3 at our disposal, and this does suffice to capture maximal systems of charts, needed to describe manifold structure. It may seem somewhat remarkable that, for example, so abstract and extensive a work as O'Neill's *Semi-Riemannian Geometry* [1983] can be translated virtually entirely without loss into a nominalistic framework, but that does seem to be the case.

Topological spaces often arise from set-theoretic constructions out of classes of functions, and one might expect that set theory is inevitably encountered. Our two final examples serve as an antidote. Consider first the construction of *product spaces* of a given family $\mathcal{F}$ of topological spaces. (We follow Kelley [1955].) Suppose that $\mathcal{F}$ is countably infinite and that each of the spaces, $(X_i, \mathcal{O}_i)$ $(i \in \mathbb{N})$ has a domain X_i at the level of a set of reals. The domain of the product space $\prod(\mathcal{F})$ consists of the Cartesian product $\prod\{X_i\}$ of the X_i together with the smallest topology for which inverses of projections onto open sets in the coordinate spaces form a subbase. That is, the domain consists of all functions f from $\mathbb{N}$ to the union of the X_i such that $f(i) \in X_i$, *i.e.*, $\prod\{X_i\} = \{f : f(i) \in X_i\}$. Since each f is (codable as) a real, this Cartesian product is at the level of a set of reals, *i.e.*, at RA^2. The *product topology* on this domain is motivated by the requirement that the projections P_i onto the factor spaces be continuous, *i.e.*, that for U open in $\mathcal{O}_i$, $P_i^{-1}[U]$ be open. The sets of this form are stipulated to be a subbase for the product topology, *i.e.*, finite intersections of such sets form a base. These are of the form $V = \{f : f(i) \in U_i \text{ for } i \in F\}$, F a finite index set (*e.g.*, a subset of $\mathbb{N}$), U_i an open set in $\mathcal{O}_i$. Such V are at the level of sets of reals, and so the standard set-theoretic construction of the least collection of these closed under arbitrary unions is available in RA^3. Note that for this construction, it is not really necessary to recognize the family $\mathcal{F}$ as an object; it suffices if one is given the collection (or whole) of the domains X_i together with the assumption that for each of these there is a collection (plurality) $\mathcal{O}_i$ of open sets of X_i; and this can be said in RA^3 on our assumption that the X_i are at the level of sets of reals. Note further that this construction in RA^3 depends on the countability of the family of given spaces. If it is uncountable, functions in the Cartesian product will be at the level of sets of reals, not reals, and the product topology itself will be a collection at RA^4. Obviously, fully general topology transcends RA^3. The same limitation does not arise, however, for products of, say, metric spaces, for which the structure is capturable at the level of relations on the points.

Finally, consider the notion of a *sheaf*, which arises as follows. One begins with holomorphic (complex differentiable) functions f, g, *etc.* on neighborhoods U, V, *etc.*, of a point c in the complex plane. f and g are said to be *germ equivalent at* c just in case f and g agree on some open neighborhood W of c such that $W \subset U \cap V$. The equivalence class $[f]_c$ of functions under this relation is called the *germ of* f *at* c. One then forms the class A_c of germs at c and then introduces $A =^{df} \bigcup_{c \in \mathbb{C}} A_c$, called *the sheaf of germs* of holomorphic functions on $\mathbb{C}$. This becomes a topological space upon taking as basic open neighborhoods of $[f]_c$ the class of all germs of f at points of the domain U of f, that is $\{[f]_c : c \in U\}$. (Neighboring germs come from the same holomorphic function.) One then considers morphisms such as the natural projection p of A onto $\mathbb{C}$ sending each germ $[f]_c$ to c, which is a local homeomorphism. Also one has a continuous function $F : A \rightarrow \mathbb{C}$ such that $F([f]_c) = f(c)$ which can be used to represent all holomorphic functions by means of cross-sections. (See Mac Lane [1986], pp. 352 f.) Now it turns out that the holomorphic functions can be coded as reals and, moreover, that germs can also be so represented *via* power series in $z - c$ convergent in some open circle about c. Thus the sheaf A becomes identifiable as a set of reals, *i.e.*, at RA^2, as do the basic open neighborhoods. The topology $\mathcal{O}$ for this space is itself then at the next level, at our familiar RA^3. Thus, even this somewhat elaborate and abstract set-theoretic construction is within the reach of the ms framework. Finally, it should be noted, all the diagrams of category theory involving these structures and the various arrows between them can be described as well, as they represent relations among finite tuples of structures (illustrating propositions which may, of course, involve universally or existentially quantified variables ranging over structures or morphisms).

It should be clear that it is not being claimed that set theory is 'never needed', whatever that might mean specifically, or that mathematics 'ought to' restrict itself to what can be nominalistically described. Rather the point has been simply to illustrate the far-reaching scope of the PA^3 and RA^3 frameworks in the interests of class-free structuralism, to give some idea of how rich a structuralism one may actually have without yet embracing anything so strong as general model theory or general category theory. All

these frameworks have their points and their places, and our task has been to understand better just what these are.

There is a sort of corollary worth noting, however, regarding scientific indispensability arguments. As Feferman has already emphasized, in connection with predicativist mathematics, it is remarkable how much of scientifically applicable mathematics can be captured within predicativist systems, and this tends to undercut Quinean arguments for set theory based on indispensability for scientific applications. (See, *e.g.*, Feferman [1992].) This holds *a fortiori* for nominalist systems as above, for these reach much further than predicativist systems, as we have already indicated. While some impredicative constructions do arise at the outer limits of applicable mathematics, it would be a real challenge to find anything in the sciences requiring mathematical power beyond the RA^3 framework. Thus, indispensability arguments should be seen in a new light, not as justifying set theory *per se*, but rather as helping (to some degree) to justify key mathematical existence assumptions such as (modal nominalistic) axioms of infinity, including not just (Ax ∞), or (Poss $\mathbb{N}$), and (Poss $\mathbb{R}$), but also the unrestricted comprehension scheme (CΣ) and related principles, such as the full comprehension principle of second-order logic. (Cf. Hellman [forthcoming].) Surprisingly perhaps, if classes are 'genuinely needed', that is probably not because of scientific applications but rather because of needs from within mathematics proper, for instance, because they allow the greatest freedom and ease of construction of anything yet devised. But whether and to what extent we have or can have evidence for the truth or possibility of models of powerful set theoretic axioms remains unresolved.

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Abstract. Recent technical developments in the logic of nominalism make it possible to improve and extend significantly the approach to mathematics developed in *Mathematics without Numbers.* After reviewing the intuitive ideas behind structuralism in general, the modal-structuralist approach as potentially class-free is contrasted broadly with other leading approaches. The machinery of nominalistic ordered pairing (Burgess-Hazen-Lewis) and plural quantification (Boolos) can then be utilized to extend the core systems of modal-structural arithmetic and analysis respectively to full, classical, polyadic third- and fourth-order number theory. The mathenatics of many structures of central importance in functional analysis, measure theory, and topology can be recovered within essentially these frameworks.